

# The homotopy type of spaces of locally convex curves in the sphere

Nicolau C. Saldanha

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## Abstract

A smooth curve  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$  is locally convex if its geodesic curvature is positive at every point. J. A. Little showed that the space of all locally convex curves  $\gamma$  with  $\gamma(0) = \gamma(1) = e_1$  and  $\gamma'(0) = \gamma'(1) = e_2$  has three connected components  $\mathcal{L}_{-1,c}$ ,  $\mathcal{L}_{+1}$ ,  $\mathcal{L}_{-1,n}$ . The space  $\mathcal{L}_{-1,c}$  is known to be contractible. In this paper we prove that  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$  are homotopy equivalent to  $\Omega\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \dots$  and  $\Omega\mathbb{S}^3 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \dots$ , respectively. We also determine the homotopy type of a family of related spaces.

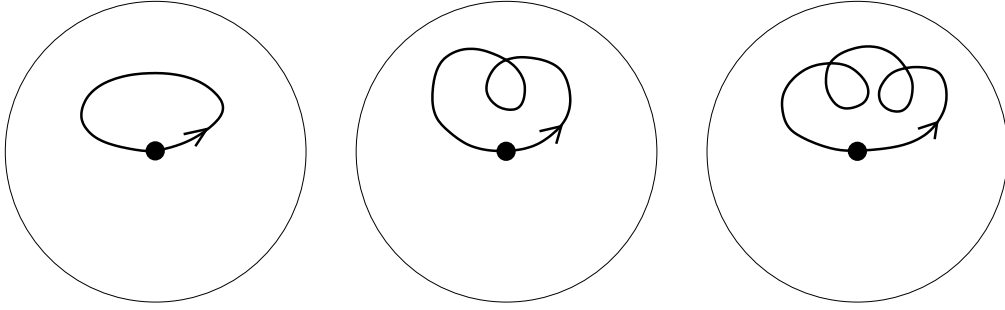
## 1 Introduction

A curve  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$  is called *locally convex* if its geodesic curvature is always positive, or, equivalently, if  $\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0$  for all  $t$ . Let  $\mathcal{L}_I$  be the space of all locally convex curves  $\gamma$  with  $\gamma(0) = \gamma(1) = e_1$  and  $\gamma'(0) = \gamma'(1) = e_2$ ; the precise topology for this space of curves will be discussed in the paper. J. A. Little [15] showed that  $\mathcal{L}_I$  has three connected components  $\mathcal{L}_{-1,c}$ ,  $\mathcal{L}_{+1}$ ,  $\mathcal{L}_{-1,n}$ ; examples of curves in each connected component are shown in Figure 1.

The connected component  $\mathcal{L}_{-1,c}$  can be defined to be the set of simple curves in  $\mathcal{L}_I$ : the space  $\mathcal{L}_{-1,c}$  is known to be contractible ([1] and [24], Lemma 5). The aim of this paper is to determine the homotopy type of the two remaining spaces  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$ . Our main result is the following.

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Figure 1: Curves in  $\mathcal{L}_{-1,c}$ ,  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$ .

**Theorem 1** *The spaces  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$  are homotopically equivalent to  $\Omega\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \dots$  and  $\Omega\mathbb{S}^3 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \mathbb{S}^{12} \vee \dots$ , respectively.*

Here  $\Omega\mathbb{S}^3$  is the space of loops in  $\mathbb{S}^3$ , i.e., the space of continuous maps  $\alpha : [0, 1] \rightarrow \mathbb{S}^3$  with  $\alpha(0) = \alpha(1) = \mathbf{1}$ , where  $\mathbf{1} \in \mathbb{S}^3$  is a base point. A more careful description of the connected components  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$  is given below.

A motivation for considering these spaces comes from differential equations. Consider the linear ODE of order 3:

$$u'''(t) + h_1(t)u'(t) + h_0(t)u(t) = 0, \quad t \in [0, 1];$$

the set of pairs of potentials  $(h_0, h_1)$  for which the equation admits 3 linearly independent periodic solutions is homotopically equivalent to  $\mathcal{L}_I$ . The corresponding problem in order 2 is much simpler ([5], [6], [21]). These spaces and variants have been discussed, among others, by B. Shapiro, M. Shapiro and B. Khesin ([23], [22]). These spaces are also the orbits of the second Gel'fand-Dikki brackets and therefore have a natural symplectic structure ([9], [10]). Furthermore, these spaces are related to the orbit classification of the Zamolodchikov Algebra ([13], [17]); these interpretations shall not be used or discussed in this paper. Although the above authors ponder about the interest of understanding the topology of such spaces, their results deal mostly with  $\pi_0$ , i.e., with counting and identifying connected components. The author has also previously proved some weaker results about the topology of these spaces, such as the fundamental group and the first few homology groups ([18], [19]); these results are now of course easy consequences of Theorem 1.

Let  $\mathcal{I}$  be the space of immersions  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$  (of class  $C^k$  for some  $k \geq 2$ ) with  $\gamma'(t) \neq 0$ ,  $\gamma(0) = e_1$ ,  $\gamma'(0) = e_2$ . Let  $\mathcal{L} \subset \mathcal{I}$  be the subspace of locally convex curves; thus, for  $\gamma \in \mathcal{I}$  we have  $\gamma \in \mathcal{L}$  if and only if  $\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0$  for all  $t$ . For each  $\gamma \in \mathcal{I}$ , consider its *Frenet frame*  $\mathfrak{F}_\gamma : [0, 1] \rightarrow SO_3$  defined by

$$\begin{pmatrix} \gamma(t) & \gamma'(t) & \gamma''(t) \end{pmatrix} = \mathfrak{F}_\gamma(t)R(t),$$

$R(t)$  being an upper triangular matrix with  $(R(t))_{11} > 0$  and  $(R(t))_{22} > 0$  (the left hand side is the  $3 \times 3$  matrix with columns  $\gamma(t)$ ,  $\gamma'(t)$  and  $\gamma''(t)$ ). In other words, the first column of  $\mathfrak{F}_\gamma(t)$  is  $\gamma(t)$ , the second is the unit tangent vector  $\mathbf{t}_\gamma(t) = \gamma'(t)/|\gamma'(t)|$  and the third column (which is now uniquely determined) is the unit normal vector  $\mathbf{n}_\gamma(t) = \gamma(t) \times \mathbf{t}_\gamma(t)$ . For  $Q \in SO_3$ , let  $\mathcal{I}_Q \subset \mathcal{I}$  be the set of curves  $\gamma \in \mathcal{I}$  for which  $\mathfrak{F}_\gamma(1) = Q$ ; similarly, let  $\mathcal{L}_Q = \mathcal{L} \cap \mathcal{I}_Q$ .

The universal (double) cover of  $SO_3$  is  $\mathbb{S}^3 \subset \mathbb{H}$ , the group of quaternions of absolute value 1; let  $\Pi : \mathbb{S}^3 \rightarrow SO_3$  be the canonical projection. For  $\gamma \in \mathcal{I}$ , the curve  $\mathfrak{F}_\gamma$  can be lifted to define  $\tilde{\mathfrak{F}}_\gamma : [0, 1] \rightarrow \mathbb{S}^3$  with  $\tilde{\mathfrak{F}}_\gamma(0) = \mathbf{1}$ ,  $\Pi \circ \tilde{\mathfrak{F}}_\gamma = \mathfrak{F}_\gamma$ . The value of  $\tilde{\mathfrak{F}}_\gamma(1)$  partitions  $\mathcal{I}_Q$  as a disjoint union  $\mathcal{I}_z \sqcup \mathcal{I}_{-z}$ : here  $\Pi(\pm z) = Q$  and  $\gamma \in \mathcal{I}_z$  if and only if  $\tilde{\mathfrak{F}}_\gamma(1) = z$ ; similarly, let  $\mathcal{L}_z = \mathcal{L}_{\Pi(z)} \cap \mathcal{I}_z$ . Notice that if  $\gamma$  is a simple curve in  $\mathcal{I}_I$  then  $\tilde{\mathfrak{F}}_\gamma(1) = -\mathbf{1}$  and therefore  $\gamma \in \mathcal{I}_{-\mathbf{1}}$ . We can now more precisely describe the three connected components of  $\mathcal{L}_I$ : the component  $\mathcal{L}_{+\mathbf{1}}$  is a special case of this definition and  $\mathcal{L}_{-\mathbf{1}}$  has two connected components  $\mathcal{L}_{-\mathbf{1},c}$  (convex or simple curves) and  $\mathcal{L}_{-\mathbf{1},n}$  (non-simple).

A locally convex curve  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$  is *convex* if  $\gamma$  intersects any geodesic (great circle) at most twice. This definition requires a couple of clarifications. First, endpoints do not count as intersections so, for instance, a simple closed locally convex curve is convex. Second, intersections are counted with multiplicity so that a tangency counts as two intersections. It follows from the definition that convex curves are simple. In this paper we will see other equivalent definitions.

A matrix  $Q \in SO_3$  is *convex* if there exists a convex arc  $\gamma \in \mathcal{L}$  with  $\mathfrak{F}_\gamma(1) = Q$ . Similarly, a quaternion  $z \in \mathbb{S}^3$  is *convex* if there exists a convex arc  $\gamma \in \mathcal{L}$  with  $\tilde{\mathfrak{F}}_\gamma(1) = z$ . It is not hard to see that if  $-z$  is convex then  $z$  is not.

We can now state a more general version of our main theorem. Here  $\mathbf{i} \in \mathbb{S}^3$  is a quaternion.

**Theorem 2** *Let  $z \in \mathbb{S}^3$ . Then the space  $\mathcal{L}_z$  is homotopically equivalent to  $\mathcal{L}_{-\mathbf{1}}$  if  $z$  is convex,  $\mathcal{L}_{\mathbf{1}}$  if  $-z$  is convex and  $\mathcal{L}_{\mathbf{i}}$  otherwise. Moreover, the following homotopy equivalences hold:*

$$\mathcal{L}_{-\mathbf{1}} \approx \Omega\mathbb{S}^3 \vee \mathbb{S}^0 \vee \mathbb{S}^4 \vee \mathbb{S}^8 \vee \cdots; \quad \mathcal{L}_{+\mathbf{1}} \approx \Omega\mathbb{S}^3 \vee \mathbb{S}^2 \vee \mathbb{S}^6 \vee \mathbb{S}^{10} \vee \cdots; \quad \mathcal{L}_{\mathbf{i}} \approx \Omega\mathbb{S}^3.$$

Let  $\Omega_z\mathbb{S}^3$  be the set of continuous curves  $\alpha : [0, 1] \rightarrow \mathbb{S}^3$ ,  $\alpha(0) = \mathbf{1}$ ,  $\alpha(1) = z$ : this is easily seen to be homeomorphic to  $\Omega\mathbb{S}^3$  and the two spaces shall from now on be identified. We just constructed the map  $\tilde{\mathfrak{F}} : \mathcal{I}_z \rightarrow \Omega_z\mathbb{S}^3$ ,  $\gamma \mapsto \tilde{\mathfrak{F}}_\gamma$ . It is a well-known fact (which follows from the Hirsch-Smale Theorem) that this map is a homotopy equivalence ([16], [12], [25]). It follows from Theorem 2 that the inclusions  $i : \mathcal{L}_z \rightarrow \mathcal{I}_z$  are homotopy equivalences only for certain quaternions  $z$ .

We now proceed to give an overview of the paper and of the proof of the main theorems. Section 2 addresses the rather technical issue of what, precisely, is the best topology for the space  $\mathcal{L}_z$ . As we shall see, we may allow for the

juxtaposition of curves (provided their Frenet frames agree); on the other hand, when desirable, we may assume curves to be smooth. The short Section 3 collects a few useful facts about the total (euclidean) curvature of spherical curves.

It follows from Little's results that a circle drawn twice and a circle drawn four times are in the same connected component of  $\mathcal{L}_I$ . In Section 4 we give a careful description of a path joining these two curves. We also prove a few facts about this path which will be needed later.

As we have just mentioned, the inclusion  $i : \mathcal{L}_z \rightarrow \mathcal{I}_z$  need not be a homotopy equivalence. In Section 5 we see that half of the story, so to speak, still holds.

**Proposition 1.1** *Let  $z \in \mathbb{S}^3$ . For any compact space  $K$  and any function  $f : K \rightarrow \mathcal{I}_z$  there exists  $g : K \rightarrow \mathcal{L}_z$  and a homotopy  $H : [0, 1] \times K \rightarrow \mathcal{I}_z$  with  $H(0, p) = f(p)$  and  $H(1, p) = g(p)$  for all  $p \in K$ .*

The maps  $i : \mathcal{L}_z \rightarrow \mathcal{I}_z$  therefore induce surjective maps  $\pi_k(\mathcal{L}_z) \rightarrow \pi_k(\mathcal{I}_z)$ .

In a nutshell, if a curve has very large (positive) geodesic curvature it looks like a phone wire and we call it *sufficiently loopy*. Any compact family  $\alpha_0 : K \rightarrow \mathcal{I}_z$  can be approximated in the  $C^0$  topology by a family  $\alpha_1 : K \rightarrow \mathcal{I}_z$  of sufficiently loopy (and therefore locally convex) curves (in Figure 2,  $\gamma_0 = \alpha_0(p)$  and  $\gamma_1 = \alpha_1(p)$  for some  $p \in K$ ). Also, families of sufficiently loopy curves can be deformed without losing the property of being locally convex.

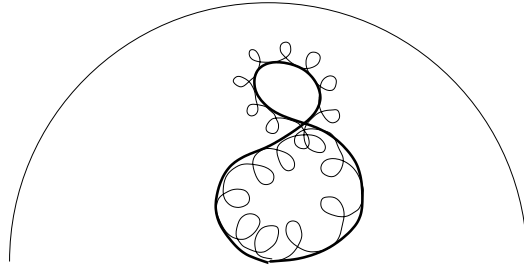


Figure 2: Curves  $\gamma_0 \in \mathcal{I}_{\pm 1}$  (thick) and  $\gamma_1 \in \mathcal{L}_{\pm 1}$  (thin).

The difficulty in proving (the false fact) that the inclusion  $\mathcal{L}_z \subset \mathcal{I}_z$  is a homotopy equivalence is that there is no *uniform* procedure to add loops to locally convex curves within the set of locally convex curves.

In Section 6 we apply the construction of Bruhat cells to our scenario. We also study projective transformations.

In Section 7 we introduce a crucial construction in our discussion: a curve  $\gamma \in \mathcal{L}_Q$  is *multiconvex* of multiplicity  $k$  if it is the juxtaposition of  $k - 1$  simple closed convex curves with a final  $k$ -th convex curve in  $\mathcal{L}_Q$  (see Figure 3).

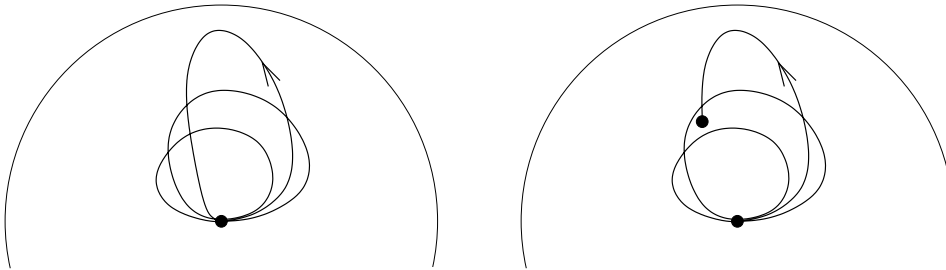


Figure 3: Two multiconvex curves of multiplicity 3

We prove in Lemma 7.1 that, given  $Q$ , the closed subset  $\mathcal{M}_k \subset \mathcal{L}_Q$  of multiconvex curves of multiplicity  $k$  is either empty or a contractible submanifold of codimension  $2k - 2$  with trivial normal bundle. Assuming  $-z$  convex, we next construct in Lemma 7.3 maps  $h_{2k-2} : \mathbb{S}^{2k-2} \rightarrow \mathcal{L}_{(-1)^{kz}}$  which intersect  $\mathcal{M}_k$  transversally and exactly once and are homotopic to a constant as maps  $\mathbb{S}^{2k-2} \rightarrow \mathcal{I}_{(-1)^{kz}}$  (details of the construction of the path in Section 4 are used here to verify that  $h_{2k-2}$  has the desired properties). Intersection with  $\mathcal{M}_k$  shows that  $h_{2k-2}$  is not homotopic to a constant as a map  $\mathbb{S}^{2k-2} \rightarrow \mathcal{L}_I$  and therefore defines a non-trivial element of the homotopy group  $\pi_{2k-2}(\mathcal{L}_{(-1)^{kz}})$  which is taken to zero by the inclusion in  $\mathcal{I}_{(-1)^{kz}}$ . Furthermore, intersection with  $\mathcal{M}_k$  defines an element of the cohomology group  $H^{2k-2}(\mathcal{L}_{(-1)^{kz}})$  not in the image of  $i^* : H^*(\mathcal{I}_{(-1)^{kz}}) \rightarrow H^*(\mathcal{L}_{(-1)^{kz}})$  (compare with [18] and [19]). This does not prove our main theorem yet but already shows that if either  $z$  or  $-z$  is convex then  $\mathcal{L}_z$  is not homotopically equivalent to  $\mathcal{I}_z$ .

In Section 8 we introduce *grafting*, a process under which loops can sometimes be added to curves. In Section 9 we define the *next step* function and learn to tell apart *good* and *bad steps*; Bruhat cells are essential here. Let  $\mathcal{Y}_z = \mathcal{L}_z \setminus \bigcup_k \mathcal{M}_k$  be the set of *complicated* (i.e., not multiconvex) curves. The new tools introduced in Sections 8 and 9 are then used in Section 10 to understand the spaces  $\mathcal{Y}_z$ .

**Proposition 1.2** *The inclusion  $\mathcal{Y}_z \subset \mathcal{I}_z$  is a weak homotopy equivalence.*

In order to determine the homotopy type of  $\mathcal{L}_z$ , start with  $\mathcal{Y}_z$  and, for each  $k$ , add the set  $\mathcal{M}_k$ . It follows from what we have proved that adding  $\mathcal{M}_k$  (if nonempty) is equivalent to attaching a sphere  $\mathbb{S}^{2k-2}$ . These properties are then sufficient to complete, in Section 11, the proof of Theorems 1 and 2.

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2005 and 2007. The author also gratefully acknowledges the support of CNPq, Capes and Faperj (Brazil).

## 2 Topology of $\mathcal{L}_Q$

We now attack a rather technical problem: defining the best topological structure for the spaces  $\mathcal{L}_Q$  and  $\mathcal{I}_Q$ .

It turns out that different topological structures obtain different spaces which are however homotopically equivalent. The  $C^k$  metric for some  $k \geq 2$  is a rather natural choice but it has the inconvenience that when constructing a homotopy we prefer not to be distracted by the necessity to smoothen out certain points of our curves (we want to be allowed, for instance, to consider a curve  $\gamma$  which is a juxtaposition of arcs of circle).

Given a smooth immersion  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$ , let  $\gamma'(t) = v_\gamma(t)\mathbf{t}_\gamma(t)$  where  $v_\gamma(t) = |\gamma'(t)| > 0$  and  $\mathbf{t}_\gamma(t)$  is the unit tangent vector to  $\gamma$ . Let  $\mathbf{n}_\gamma(t) = \gamma(t) \times \mathbf{t}_\gamma(t)$  be the unit normal vector to  $\gamma$ , so that

$$\mathbf{t}'_\gamma(t) = -v_\gamma(t)\gamma(t) + \hat{v}_\gamma(t)\mathbf{n}_\gamma(t), \quad \mathbf{n}'_\gamma(t) = -\hat{v}_\gamma(t)\mathbf{t}_\gamma(t)$$

where  $\hat{v}_\gamma(t) = \kappa_\gamma(t)v_\gamma(t)$ ,  $\kappa_\gamma(t)$  being the geodesic curvature of  $\gamma$ . The curve  $\gamma$  is locally convex if and only if  $\hat{v}_\gamma(t) > 0$  for all  $t$ . In matrix notation,  $\gamma(t)$ ,  $\mathbf{t}_\gamma(t)$  and  $\mathbf{n}_\gamma(t)$  are the columns of the orthogonal matrix  $\mathfrak{F}_\gamma(t)$  which satisfies

$$\mathfrak{F}'_\gamma(t) = \mathfrak{F}_\gamma(t)\Lambda_\gamma(t); \quad \Lambda_\gamma(t) = \begin{pmatrix} 0 & -v_\gamma(t) & 0 \\ v_\gamma(t) & 0 & -\hat{v}_\gamma(t) \\ 0 & \hat{v}_\gamma(t) & 0 \end{pmatrix}.$$

Let  $V \subset so_3$  be the plane (i.e., 2-dimensional real vector space) of matrices  $M$  with  $(M)_{31} = 0$ ;  $\Lambda_\gamma$  can be considered a function from  $[0, 1]$  to  $V$ . Let  $V_{\mathcal{I}} \subset V$  be the half-plane  $(M)_{21} > 0$ ;  $\Lambda_\gamma$  can also be considered a function  $\Lambda_\gamma : [0, 1] \rightarrow V_{\mathcal{I}}$ . Conversely, given a smooth function  $\Lambda_\gamma : [0, 1] \rightarrow V_{\mathcal{I}}$  or, equivalently,  $v_\gamma$  and  $\hat{v}_\gamma$ , the above equations together with  $\mathfrak{F}_\gamma(0) = I$  may be interpreted as an initial value problem defining  $\mathfrak{F}_\gamma(t)$  and therefore  $\gamma(t) = \mathfrak{F}_\gamma(t)e_1$ .

Our aim is to consider a reasonably large space of functions which still allows the initial value problem above to be solved. The Hilbert space  $L^2([0, 1])$  is now a natural choice: for  $v, \hat{v} \in L^2([0, 1])$  the initial value problem can be solved. We therefore interpret  $\mathcal{I}_Q$  to be the closed subset of  $(L^2([0, 1]))^2$  of pairs of functions  $(w, \hat{v})$  such that the solution  $\Gamma : [0, 1] \rightarrow SO_3$  of the initial value problem

$$\Gamma'(t) = \Gamma(t)\Lambda(t), \quad \Gamma(0) = I, \tag{1}$$

satisfies  $\Gamma(1) = Q$ , where

$$\Lambda(t) = \begin{pmatrix} 0 & -v(t) & 0 \\ v(t) & 0 & -\hat{v}(t) \\ 0 & \hat{v}(t) & 0 \end{pmatrix}, \quad v(t) = \frac{w(t) + \sqrt{(w(t))^2 + 4}}{2}$$

(we have  $w(t) = v(t) - 1/v(t)$ ,  $v(t) > 0$ ). The fact that  $\Lambda(t)$  belongs to the Lie algebra  $so_3$  guarantees that  $\Gamma(t)$  assumes values in the corresponding Lie group  $SO_3$ . Also, the function  $\Gamma$  defined above is continuous, absolutely continuous and differentiable almost everywhere. In this way  $\mathcal{I}_Q \subset (L^2([0, 1]))^2$  is a smooth Hilbert manifold of codimension 3. Indeed, consider the map  $\omega_{\mathcal{I}} : (L^2([0, 1]))^2 \rightarrow SO_3$  taking  $(w, \hat{v})$  to  $\Gamma(1)$ , where  $\Gamma : [0, 1] \rightarrow SO_3$  is defined by the initial value problem (1) above. Smooth dependence on parameters tells us that this map is smooth; let us compute its derivative.

In general, for a curve  $\Gamma : [0, 1] \rightarrow SO_3$ , write  $\Gamma(t_0; t_1) = (\Gamma(t_0))^{-1}\Gamma(t_1)$ . Let  $L$  be a one-parameter family of functions  $L(s) : [0, 1] \rightarrow V_{\mathcal{I}}$ ,  $s \in (-\epsilon, \epsilon)$  with  $L(0) = \Lambda$ . Let  $G(s) : [0, 1] \rightarrow SO_3$  be the solution of

$$(G(s))'(t) = (G(s))(t)(L(s))(t), \quad (G(s))(0) = I$$

so that  $G(0) = \Gamma$ ; notice that the derivative in this initial value problem is with respect to  $t$ . An easy computation gives that the derivative of  $G$  (with respect to  $s$ ) satisfies

$$(G'(s))(t)(G(s)(t))^{-1} = \int_0^t (G(s))(\tau)(L'(s))(\tau)((G(s))(\tau))^{-1} d\tau.$$

Assume, for instance, that  $(L'(0))(t)$  consists of three smooth narrow bumps around times  $t_i$  so that

$$(\Gamma(1))^{-1}(G'(0))(t) \approx \sum_i (\Gamma(t_i; 1))^{-1}(L'(0))(t_i)\Gamma(t_i; 1).$$

An easy computation shows that the spaces  $(\Gamma(t_i; 1))^{-1}V\Gamma(t_i; 1) \subset so_3$  are not constant and therefore  $(\Gamma(1))^{-1}(G'(0))(t)$  may assume any value in  $so_3$ . The derivative of  $\omega_{\mathcal{I}}$  is therefore surjective. The map  $\omega_{\mathcal{I}}$  is thus a submersion and  $\mathcal{I}_Q$  a regular level set. The geometric description of  $\mathcal{I}_Q$  comes from the identification  $(w, \hat{v}) \leftrightarrow \gamma$ , where  $\gamma(t) = \Gamma(t)e_1$ .

Similarly, let  $V_{\mathcal{L}} \subset V \subset so_3$  be the quarter-plane  $(M)_{31} = 0$ ,  $(M)_{32} > 0$ ,  $(M)_{21} > 0$ . If  $\gamma$  is locally convex then the image of  $\Lambda_{\gamma}$  is contained in  $V_{\mathcal{L}}$  and, conversely, given a smooth function  $\Lambda : [0, 1] \rightarrow V_{\mathcal{L}}$ , equation (1) obtains a locally convex curve. Define  $\mathcal{L}_Q \subset (L^2([0, 1]))^2$  to be the set of pairs  $(w, \hat{w})$  such that  $(w, \hat{v}) \in \mathcal{I}_Q$  where

$$\hat{v}(t) = \frac{\hat{w}(t) + \sqrt{(\hat{w}(t))^2 + 4}}{2}.$$

As above, define  $\omega : (L^2([0, 1]))^2 \rightarrow SO_3$  taking  $(w, \hat{w})$  to  $\Gamma(1)$ , where  $\Gamma$  is again defined by equation (1). The computation above also shows that the smooth map  $\omega$  is a submersion and  $\mathcal{L}_Q$  is a smooth Hilbert submanifold of codimension 3 in  $(L^2([0, 1]))^2$ . We have a natural injective map  $\mathcal{L}_Q \rightarrow \mathcal{I}_Q$  taking  $(w, \hat{w})$  to  $(w, \hat{v})$ . In the spirit of considering  $\mathcal{L}_Q$  and  $\mathcal{I}_Q$  to be sets of curves we call this map an inclusion (even though it is not an isometry with the above metric).

The space  $\mathcal{L}_Q$  we just defined is rather large. The space  $\mathcal{L}_Q^{[C^k]}$  of locally convex curves of class  $C^k$  ( $k \geq 2$ ) is now naturally identified to a dense subset  $\mathcal{L}_Q^{[C^k]} \subset \mathcal{L}_Q$  with a different topology. But are these two spaces similar? Or, more precisely, is the inclusion a weak homotopy equivalence? The answer here is yes.

In order to see this, first consider  $\mathcal{L}_Q^{[[C^k]]}$ ,  $k \geq 0$ , the subset of  $(C^k([0, 1]))^2$  of pairs  $(w, \hat{w})$  such that  $\Gamma(1) = Q$  (where  $\Gamma$  is defined by the initial value problem (1)). The inclusion  $\mathcal{L}_Q^{[[C^k]]} \subset \mathcal{L}_Q$  is a homotopy equivalence: this follows directly from Theorem 2 from [4]. For the convenience of the reader, we quote here a simplified version of that result.

**Proposition 2.1** *Let  $X$  and  $Y$  be separable Banach spaces. Suppose  $i : Y \rightarrow X$  is a bounded, injective linear map with dense image and  $M \subset X$  a smooth, closed submanifold of finite codimension. Then  $N = i^{-1}(M)$  is a smooth closed submanifold of  $Y$  and the restriction  $i : N \rightarrow M$  is a homotopy equivalence.*

For  $k \geq 1$  the curves  $\gamma$  are now of class  $C^2$  and we may assume them to be parametrized by a constant multiple of arc length (this does not change the homotopy type of the space since the group of orientation-preserving diffeomorphisms of  $[0, 1]$  is contractible). In terms of the pairs  $(w, \hat{w})$ , this says that we may assume  $w$  to be constant. But for  $w$  constant,  $\hat{w}$  is of class  $C^k$  if and only if  $\gamma$  is of class  $C^{k+2}$ , completing the argument.

As a consequence, we may assume our curves  $\gamma$  to be as smooth as our constructions require but when constructing a homotopy we may use curves for which the curvature is only piecewise continuous without further ado.

### 3 Total curvature

Given a locally convex curve  $\gamma : [a, b] \rightarrow \mathbb{S}^2$ , let  $\kappa_\gamma : [a, b] \rightarrow \mathbb{R}$  be the geodesic curvature of  $\gamma$ . This must not be confused with the euclidean curvature  $\kappa_\gamma^E$  of  $\gamma$  interpreted as a curve in  $\mathbb{R}^3$ :  $\kappa_\gamma^E(t) = \sqrt{1 + \kappa_\gamma^2(t)}$ . The total (euclidean) curvature of  $\gamma : [a, b] \rightarrow \mathbb{S}^2$  is

$$\text{tot}(\gamma) = \int_{[a, b]} \kappa_\gamma^E(t) |\gamma'(t)| dt.$$



**Lemma 3.1** *Let  $\gamma : [a, b] \rightarrow \mathbb{S}^2$  be a locally convex curve.*

(a) *The total curvature of  $\gamma$  equals twice the length of  $\tilde{\mathfrak{F}}_\gamma$ :*

$$\text{tot}(\gamma) = 2 \int_{[a,b]} |\tilde{\mathfrak{F}}'_\gamma(t)| dt.$$

(b) *If  $\gamma \in \mathcal{L}_I$  is a closed convex curve then  $\text{tot}(\gamma) \in [2\pi, 4\pi]$ .*

(c) *If  $\gamma \in \mathcal{L}$  is a convex arc then  $\text{tot}(\gamma) \in (0, 4\pi]$ .*

**Proof:** Item (a) is a straightforward computation. The first inequality in item (b) is the well known general fact that the total curvature of a closed curve is at least  $2\pi$ . Alternatively, given (a),  $\tilde{\Gamma} = \tilde{\mathfrak{F}}_\gamma : [0, 1] \rightarrow \mathbb{S}^3$  satisfies  $\tilde{\Gamma}(0) = \mathbf{1}$ ,  $\tilde{\Gamma}(1) = -\mathbf{1}$  so of course the length of  $\tilde{\Gamma}$  must be at least  $\pi$ . For the second inequality in (b), first notice that  $\kappa_\gamma^E(t) \leq 1 + \kappa_\gamma(t)$  so we must prove that

$$I_1 + I_2 < 4\pi, \quad I_1 = \int_{[0,1]} |\gamma'(t)| dt, \quad I_2 = \int_{[0,1]} \kappa_\gamma(t) |\gamma'(t)| dt.$$

By Gauss-Bonnet, the  $I_2 = 2\pi - A_\gamma < 2\pi$ , where  $A_\gamma$  is the area of the smaller disk in  $\mathbb{S}^2$  bounded by  $\gamma$ . Let  $\mathbf{n}_\gamma : [0, 1] \rightarrow \mathbb{S}^2$  be the unit normal vector:  $\mathbf{n}_\gamma(t) = \tilde{\mathfrak{F}}_\gamma(t)e_3$ . A straightforward computation shows that

$$|\mathbf{n}'_\gamma(t)| = \kappa_\gamma(t) |\gamma'(t)|, \quad \kappa_{\mathbf{n}_\gamma}(t) = 1/\kappa_\gamma(t).$$

Thus, again by Gauss-Bonnet,

$$I_1 = \int_{[0,1]} \kappa_{\mathbf{n}_\gamma}(t) |\mathbf{n}'_\gamma(t)| dt = 2\pi - A_{\mathbf{n}_\gamma} < 2\pi$$

where  $A_{\mathbf{n}_\gamma}$  is (of course) the area of the smaller disk in  $\mathbb{S}^2$  bounded by  $\mathbf{n}_\gamma$ . This completes the proof of (b).

For item (c), consider a convex arc  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$ . For arbitrarily small  $\epsilon > 0$ ,  $\gamma|_{[0, 1-\epsilon]}$  can be extended to a closed convex curve  $\tilde{\gamma}$ . Thus, from (b),  $\text{tot}(\gamma|_{[0, 1-\epsilon]}) < \text{tot}(\tilde{\gamma}) < 4\pi$ . Since this estimate holds for all  $\epsilon$ ,  $\text{tot}(\gamma) \leq 4\pi$ . ■

## 4 Connected components of $\mathcal{L}_I$

Recall that Little showed that  $\mathcal{L}_I$  has three connected components  $\mathcal{L}_{-1,c}$ ,  $\mathcal{L}_1$  and  $\mathcal{L}_{-1,n}$  (see Figure 1). In this section we review some of the relevant constructions in his proof: we do this also in order to introduce objects which will come up again in later sections.

Let  $\mathcal{C}_\nu$  be the circle with diameter  $e_1 e_3$ , parametrized by  $\nu_1 \in \mathcal{L}_I$ ,

$$\nu_1(t) = \frac{1}{2} \left( 1 + \cos(2\pi t), \sqrt{2} \sin(2\pi t), 1 - \cos(2\pi t) \right)$$

for which

$$\tilde{\mathfrak{F}}_{\nu_1}(t) = \exp \left( \pi t \hat{\mathbf{k}} \right), \quad \tilde{\Lambda}_{\nu_1}(t) = \pi \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} = \frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}},$$

where  $\tilde{\Lambda}_\gamma(t) = (\tilde{\mathfrak{F}}_\gamma(t))^{-1} \tilde{\mathfrak{F}}'_\gamma(t)$ . For positive real number  $s$ , let  $\nu_s(t) = \nu_1(st)$  so that  $\nu_s \in \mathcal{L}_{\exp(\pi s \hat{\mathbf{k}})}$ . In particular, for integer  $k > 0$ ,  $\nu_k \in \mathcal{L}_{(-1)^k}$ . We also have  $\nu_1 \in \mathcal{L}_{-1,c}$  and  $\nu_k \in \mathcal{L}_{-1,n}$  for  $k$  odd,  $k > 1$ . Notice that  $\text{tot}(\nu_s) = 2\pi s$ .

It follows from Little's theorem that  $\nu_2$  and  $\nu_4$  are in the same connected component  $\mathcal{L}_{+1}$  of  $\mathcal{L}_I$ . Indeed, a path from  $\nu_2$  to  $\nu_4$  is shown in Figure 4. The passage from the fourth to the fifth figure is the only one where it is important to recall that we are in a sphere, not in the plane: one way to think of this is that the triangle became large and the bulk of the sphere passed through the triangle. Notice that the geodesic curvature remains positive throughout the process.

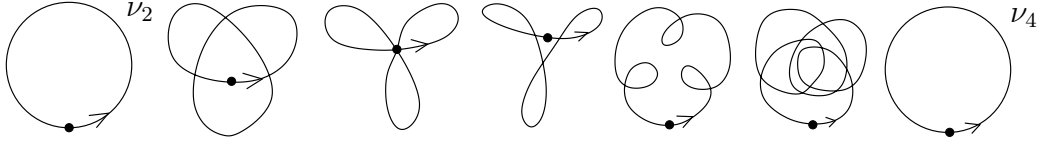


Figure 4: A path from  $\nu_2$  to  $\nu_4$ .

We now present an explicit construction of a family of such curves. Consider the equator of the unit sphere  $\mathbb{S}^2$  contained in the plane  $z = 0$ . Fix a unit normal vector  $N = (0, 0, 1)$  to the plane and call it *up*. Consider six equally spaced points  $P_0 = P_3 = (1/2, -\sqrt{3}/2, 0)$ ,  $Q_0 = Q_3 = (1, 0, 0)$ ,  $P_1 = (1/2, \sqrt{3}/2, 0)$ ,  $Q_1 = -P_0$ ,  $P_2 = -Q_0$  and  $Q_2 = -P_1$  along the equator. Notice that  $Q_i/2$  is the midpoint of the segment  $P_i P_{i+1}$ . For  $\alpha \in \mathbb{R}$ , let  $Q_i^+(\alpha)$  be the only point on the unit sphere such that the vector  $Q_i^+(\alpha) - (Q_i/2)$  is a positive multiple of  $(\cos \alpha)Q_i + (\sin \alpha)N$ ; let  $Q_i^-(\alpha) = Q_i^+(-\alpha)$ . Thus, the function  $Q_i^+$  parametrizes the circle passing through  $\pm Q_i$  and  $\pm N$ . Let  $\mathcal{C}_i^\pm(\alpha)$  be the circle containing  $P_i$ ,  $Q_i^\pm(\alpha)$  and  $P_{i+1}$  and let  $A_i^\pm(\alpha) \subset \mathcal{C}_i^\pm(\alpha)$  be the arc from  $P_i$  through  $Q_i^\pm(\alpha)$  to  $P_{i+1}$ .

Orient the circles  $\mathcal{C}_i^\pm(\alpha)$  and the arcs  $A_i^\pm(\alpha)$  from  $P_i$  through  $Q_i^\pm(\alpha)$  to  $P_{i+1}$ ; thus, for  $\alpha \in (0, \pi)$ ,  $\mathcal{C}_i^+(\alpha)$  (resp.  $\mathcal{C}_i^-(\alpha)$ ) has negative (resp. positive) geodesic curvature. Parametrize the arcs  $A_0^+(\alpha)$ ,  $A_1^-(\alpha)$ ,  $A_2^+(\alpha)$ ,  $A_3^-(\alpha)$ ,  $A_4^+(\alpha)$ ,  $A_5^-(\alpha)$  and  $A_6^+(\alpha)$  by a multiple of arc length using the domains  $[-1/12, 1/12]$ ,  $[1/12, 3/12]$ ,  $[3/12, 5/12]$ ,  $[5/12, 7/12]$ ,  $[7/12, 9/12]$ ,  $[9/12, 11/12]$  and  $[11/12, 13/12]$ , respectively. Concatenate the above parametrizations to define a parametrization  $\beta_\alpha : [0, 1] \rightarrow \mathbb{S}^2$  by a multiple of arc length of a curve  $C_\alpha \subset \mathbb{S}^2$ . In particular,

$\beta_\alpha(0) = Q_0^+(\alpha)$ ,  $\beta_\alpha(1/12) = P_1$ ,  $\beta_\alpha(2/12) = Q_1^-(\alpha)$  and so on. Notice that the curve  $\beta_\alpha$  is of class  $C^1$ , even at the points  $t = j/12$ . The curve  $\beta_\alpha$  is an immersion and is not locally convex. Define  $B_\alpha = \mathfrak{F}_{\beta_\alpha} : [0, 1] \rightarrow SO_3$  as usual; lift this to define  $\tilde{B}_\alpha : [0, 1] \rightarrow \mathbb{S}^3$  with  $\tilde{B}_0(0) = \mathbf{1}$  and  $\tilde{B}_\alpha(t)$  a continuous function of  $\alpha$  and  $t$ .

Define  $\tilde{\Gamma}_\alpha(t) = \tilde{B}_\alpha(t) \exp(\pi \mathbf{j}/8)$ ,  $\Gamma_\alpha = \Pi \circ \tilde{\Gamma}_\alpha$  and (of course)  $\gamma_\alpha(t) = \Gamma_\alpha(t)e_1$ . Geometrically,  $\gamma_\alpha$  is the concatenation of six circle arcs  $\tilde{A}_i^\pm$ , obtained from  $A_i^\pm$  by increasing or decreasing the radius by  $\pi/4$ . A simple computation shows that the geodesic curvature of  $\gamma_\alpha$  is always positive (the minimum and maximum values for the curvature are  $2 - \sqrt{3}$  and  $2 + \sqrt{3}$ , both obtained for  $\alpha = \pi/2$ ). Furthermore,  $\text{tot}(\gamma_\alpha)$  is a strictly increasing function of  $\alpha \in [0, \pi]$  with  $\text{tot}(\gamma_0) = 4\pi$ ,  $\text{tot}(\gamma_\pi) = 8\pi$ .

The following result is due to Little ([15]); we state and prove it here in order to get used to the above construction which shall be needed later.

**Lemma 4.1** *Let  $n$  be a positive integer,  $n > 1$ . The curves  $\nu_n$  and  $\nu_{n+2}$  are in the same connected component of  $\mathcal{L}_I$ .*

**Proof:** Define  $f : [0, 1] \rightarrow \mathcal{L}_I$  by  $f(s)(t) = (\Gamma_{\pi s}(0))^{-1} \gamma_{\pi s}(t)$ . We have  $f(0) = \nu_2$  and  $f(1) = \nu_4$ , completing the proof for the case  $n = 2$ . For larger  $n$  just consider  $\nu_n$  as a concatenation of  $\nu_{n-2}$  with  $\nu_2$ , keep  $\nu_{n-2}$  fixed and apply the above construction to move from  $\nu_2$  to  $\nu_4$ , thus obtaining a path in  $\mathcal{L}_I$  from  $\nu_n$  to  $\nu_{n+2}$ .  $\blacksquare$

We prove an auxiliary result concerning the curves  $\beta_\alpha$  and  $\gamma_\alpha$  for later use. A *common oriented tangent* to two oriented circles is an oriented geodesic which is tangent to both circles, with compatible orientation at tangency points. Thus, for instance,  $\mathcal{C}_i^\pm(\alpha)$  and  $\mathcal{C}_{i+1}^\mp(\alpha)$  are tangent at  $P_{i+1}$ ; they also have a common oriented tangent, a geodesic passing through  $P_{i+1}$ . Let  $\mathcal{C}_\mathbf{k}$  (resp.  $\mathcal{C}_\mathbf{k}$ ) be the great circle and subgroup of  $\mathbb{S}^3$  of points of the form  $\exp(s\mathbf{k})$  (resp.  $\exp(s\hat{\mathbf{k}})$ ),  $s \in \mathbb{R}$ . Recall that we write  $\tilde{B}_\alpha(t_0; t_1) = (\tilde{B}_\alpha(t_0))^{-1} \tilde{B}_\alpha(t_1)$ .

**Lemma 4.2** *Let  $\alpha \in (0, \pi)$ .*

- (a) *The common oriented tangents between two distinct circles among  $\mathcal{C}_i^\pm(\alpha)$ ,  $i = 0, 1, 2$ , are only the geodesics passing through the tangency point  $P_{i+1}$  between  $\mathcal{C}_i^\pm(\alpha)$  and  $\mathcal{C}_{i+1}^\mp(\alpha)$ .*
- (b) *For  $t_0, t_1 \in [0, 1)$ , if  $\tilde{B}_\alpha(t_0; t_1) \in \mathcal{C}_\mathbf{k}$  then  $t_0 = t_1$ .*
- (c) *For  $t_0, t_1 \in [0, 1)$ , if  $\tilde{\Gamma}_\alpha(t_0; t_1) \in \mathcal{C}_\mathbf{k}$  then  $t_0 = t_1$ .*

**Proof:** Recall that our circles  $\mathcal{C}_i^\pm(\alpha)$  are not geodesics. Each circle therefore defines a disk (the smaller connected component of the complement).

For item (a), we have three essentially different pairs of circles to consider:  $(\mathcal{C}_0^+(\alpha), \mathcal{C}_0^-(\alpha))$ ,  $(\mathcal{C}_0^+(\alpha), \mathcal{C}_1^+(\alpha))$  and  $(\mathcal{C}_0^+(\alpha), \mathcal{C}_1^-(\alpha))$ . In the first case, the two circles have opposite orientations and therefore the circles and corresponding disks would have to lie on opposite sides of a common oriented tangent. Since the open disks intersect, no common oriented tangent exists. In the second case orientations agree and therefore both disks would lie on the same side of a common oriented tangent. But the union of the two closed disks contains the arc from  $P_0$  through  $P_1$  to  $P_2$  and therefore is not contained in a hemisphere. Thus also in this case no common oriented tangent exists. Finally, in the third case again the two circles have opposite orientations and therefore the disks must lie on opposite sides of a common oriented tangent. But the closed disks touch at  $P_1$ : the common oriented tangent must therefore pass through this point, completing the proof of the first claim.

For item (b), assume by contradiction that  $\tilde{B}_\alpha(t_0; t_1) = \exp(s\mathbf{k})$ ,  $t_0 \neq t_1$ . Consider the curve  $\tilde{\Gamma} : [0, 1] \rightarrow \mathbb{S}^3$  given by  $\tilde{\Gamma}(t) = \tilde{B}_\alpha(t_0) \exp(2\pi t \mathbf{k})$ . Notice that  $\tilde{\Gamma}(0) = \tilde{B}_\alpha(t_0)$  and  $\tilde{\Gamma}(s/(2\pi)) = \tilde{B}_\alpha(t_1)$ . Also  $\tilde{\Gamma}(t) = \tilde{\mathfrak{F}}_\gamma(t)$  for  $\gamma$  the geodesic  $\gamma(t) = B_\alpha(t_0)(\cos(4\pi t), \sin(4\pi t), 0)$ . Thus  $\gamma$  is an oriented tangent to the curve  $C_\alpha$  at two distinct points. These two points must belong to different circles  $\mathcal{C}_i^\pm$  for a circle can not be twice tangent to the same geodesic. But this contradicts item (a).

Finally, for item (c), notice that

$$\tilde{\Gamma}_\alpha(t_0; t_1) = \exp(-\pi \mathbf{j}/8) \tilde{B}_\alpha(t_0; t_1) \exp(\pi \mathbf{j}/8)$$

and therefore  $\tilde{\Gamma}_\alpha(t_0; t_1) = \exp(s\hat{\mathbf{k}})$  implies

$$\begin{aligned} \tilde{B}_\alpha(t_0; t_1) &= \exp(\pi \mathbf{j}/8) \exp(s\hat{\mathbf{k}}) \exp(-\pi \mathbf{j}/8) \\ &= \exp\left(s \exp(\pi \mathbf{j}/8) \hat{\mathbf{k}} \exp(-\pi \mathbf{j}/8)\right) = \exp(s\mathbf{k}). \end{aligned}$$

Thus item (b) implies item (c). ■

## 5 Adding loops

In this section we present a few facts related to adding loops to a curve (or family of curves), including the proof of Proposition 1.1. This is of course similar to the proof of the Hirsch-Smale theorem ([12], [25]). The reader familiar with Gromov's ideas will also recognize this as an easy instance of the h-principle ([7], [11]); others will be reminded of Thurston's method for performing the sphere eversion by corrugations ([14]).

We need a precise version for the notion of adding  $n$  loops at a point  $t_0$  of a curve  $\gamma$ . For  $\gamma \in \mathcal{I}$ ,  $t_0 \in (0, 1)$  and  $n$  a positive integer let  $\gamma^{[t_0 \# n]} \in \mathcal{I}$  be defined by

$$\gamma^{[t_0 \# n]}(t) = \begin{cases} \gamma(t), & 0 \leq t \leq t_0 - 2\epsilon; \\ \gamma(2t - t_0 + 2\epsilon), & t_0 - 2\epsilon \leq t \leq t_0 - \epsilon; \\ \mathfrak{F}_\gamma(t_0)\nu_n\left(\frac{t-t_0+\epsilon}{2\epsilon}\right), & t_0 - \epsilon \leq t \leq t_0 + \epsilon; \\ \gamma(2t - t_0 + 2\epsilon), & t_0 + \epsilon \leq t \leq t_0 + 2\epsilon; \\ \gamma(t), & t_0 + 2\epsilon \leq t \leq 1; \end{cases}$$

here  $\epsilon > 0$  is taken sufficiently small. In words, we follow  $\gamma$  normally almost until  $t_0$ : we then run a little in order to have room to insert  $\nu_n$  (appropriately moved to the correct position), run a little again and then continue as  $\gamma$ . Since reparametrizations of curves are not particularly interesting (the group of orientation-preserving diffeomorphisms of  $[0, 1]$  is contractible) the precise value of  $\epsilon$  is not particularly interesting either. Notice that if  $\gamma \in \mathcal{I}_z$  then  $\gamma^{[t_0 \# n]} \in \mathcal{I}_{(-1)^{n_z}}$ . Also, if  $\gamma$  is locally convex then so is  $\gamma^{[t_0 \# n]}$ , with  $\text{tot}(\gamma^{[t_0 \# n]}) = 2\pi n + \text{tot}(\gamma)$ . We use the notation  $\gamma^{[t_0 \# n_0; t_1 \# n_1]}$  for  $(\gamma^{[t_0 \# n_0]})^{[t_1 \# n_1]} = (\gamma^{[t_1 \# n_1]})^{[t_0 \# n_0]}$ . Also, given  $t_0 : K \rightarrow (0, 1)$  and  $f : K \rightarrow \mathcal{I}_Q$  continuous functions, let  $f^{[t_0 \# n]} : K \rightarrow \mathcal{I}_Q$  be defined by  $f^{[t_0 \# n]}(p) = (f(p))^{[t_0(p) \# n]}$ .

Notice that in  $\mathcal{I}_Q$  it is easy to introduce a pair of loops at any point of the curve.

**Lemma 5.1** *Let  $K$  be a compact set,  $Q \in SO_3$  and  $n$  a positive even integer. Let  $t_0 : K \rightarrow (0, 1)$  and  $f : K \rightarrow \mathcal{I}_Q$  be continuous functions. Then  $f$  and  $f^{[t_0 \# n]}$  are homotopic.*



Figure 5: How to add two small loops to a curve in  $\mathcal{I}_Q$ .

**Proof:** The process is illustrated in Figure 5; in the final step one of the loops becomes big, goes around the sphere and shrinks again. We do not think that an explicit formula is helpful.  $\blacksquare$

Somewhat similarly, for locally convex curves we have the following.

**Lemma 5.2** *Let  $K$  be a compact set,  $Q \in SO_3$  and  $n > 1$  an integer. Let  $t_0 : K \rightarrow (0, 1)$  and  $f : K \rightarrow \mathcal{L}_Q$  be continuous functions. Then  $f^{[t_0 \# n]}$  and  $f^{[t_0 \# (n+2)]}$  are homotopic, i.e., there exists  $H : [0, 1] \times K \rightarrow \mathcal{L}_Q$  with  $H(0, p) = f^{[t_0 \# n]}(p)$ ,  $H(1, p) = f^{[t_0 \# (n+2)]}(p)$ . We may furthermore assume that  $2\pi n + \text{tot}(f(p)) \leq \text{tot}(H(s, p)) \leq 2\pi(n+2) + \text{tot}(f(p))$ .*

**Proof:** We use the path from  $\nu_n$  to  $\nu_{n+2}$  in  $\mathcal{L}_I$  constructed in Lemma 4.1 to modify the curve  $f(p)$  in the interval  $[t_0(p) - \epsilon, t_0(p) + \epsilon]$  only.  $\blacksquare$

The above statement actually holds for  $n = 1$  (by borrowing a little elbow room from the surrounding curve) but we do not need this observation. On the other hand, the statement critically fails for  $n = 0$ . Indeed, we know (from Fenchel, Little and others; [8], [15]) that  $\nu_1$  and  $\nu_3$  are not in the same connected component of  $\mathcal{L}_I$ .

We now need a construction corresponding to spreading loops along the curve, as in Figure 2. For  $\gamma \in \mathcal{I}$  and  $n > 0$ , define  $(F_n(\gamma))(t) = \mathfrak{F}_\gamma(t)\nu_n(t)$ . Notice that if  $\gamma \in \mathcal{I}_z$  then  $F_n(\gamma) \in \mathcal{I}_{(-1)^n z}$ .

**Lemma 5.3** *Let  $K$  be a compact set and let  $f : K \rightarrow \mathcal{I}_z$  be a continuous function. Then, for sufficiently large  $n$ , the image of  $F_{2n} \circ f$  is contained in  $\mathcal{L}_z$ .*

**Proof:** Let  $C > 1$  be a constant such that  $|(\mathfrak{F}_\gamma)'(t)| < C$  and  $|(\mathfrak{F}_\gamma)''(t)| < C$  for any  $\gamma = f(p)$ ,  $p \in K$  and for any  $t \in [0, 1]$  (we use here the invariant metric on  $SO_3$  inherited from  $\mathbb{R}^9$ ). Let  $\epsilon > 0$  be such that if  $|v_1 - \nu_1'(t)| < \epsilon$  and  $|v_2 - \nu_1''(t)| < \epsilon$  then  $\det(\nu_1(t), v_1, v_2) > 0$ . Take  $n > 20C/\epsilon$ .

For  $\gamma = f(p)$ , write

$$\tilde{\gamma}(t) = (F_{2n}\gamma)(t) = \mathfrak{F}_\gamma(t)\nu_{2n}(t) = \mathfrak{F}_\gamma(t)\nu_1(2nt)$$

so that

$$\begin{aligned}\tilde{\gamma}'(t) &= \mathfrak{F}_\gamma'(t)\nu_1(2nt) + 2n\mathfrak{F}_\gamma(t)\nu_1'(2nt) \\ \tilde{\gamma}''(t) &= \mathfrak{F}_\gamma''(t)\nu_1(2nt) + 4n\mathfrak{F}_\gamma'(t)\nu_1'(2nt) + 4n^2\mathfrak{F}_\gamma(t)\nu_1''(2nt)\end{aligned}$$

and therefore, after a few manipulations,

$$\left| \frac{\tilde{\gamma}'(t)}{2n} - \mathfrak{F}_\gamma(t)\nu_1'(2nt) \right| < \epsilon, \quad \left| \frac{\tilde{\gamma}''(t)}{4n^2} - \mathfrak{F}_\gamma(t)\nu_1''(2nt) \right| < \epsilon$$

or, equivalently,

$$\left| \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}'(t)}{2n} - \nu_1'(2nt) \right| < \epsilon, \quad \left| \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}''(t)}{4n^2} - \nu_1''(2nt) \right| < \epsilon.$$

It follows that

$$\det \left( \nu_1(2nt), \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}'(t)}{2n}, \frac{(\mathfrak{F}_\gamma(t))^{-1}\tilde{\gamma}''(t)}{4n^2} \right) > 0$$

and therefore that  $\det(\tilde{\gamma}(t), \tilde{\gamma}'(t), \tilde{\gamma}''(t)) > 0$ , which is what we needed.  $\blacksquare$

**Lemma 5.4** *Let  $K$  be a compact set,  $f : K \rightarrow \mathcal{I}_z$  and  $t_0 : K \rightarrow (0, 1)$  continuous maps. Then  $F_{2n} \circ f$  is homotopic to  $f^{[t_0 \# (2n)]}$ , i.e., there exists  $H : [0, 1] \times K \rightarrow \mathcal{I}_z$  such that  $H(0, \cdot) = F_{2n} \circ f$  and  $H(1, \cdot) = f^{[t_0 \# (2n)]}$ . Furthermore, if the image of  $f$  is contained in  $\mathcal{L}_z$  then, for sufficiently large  $n$ , the image of  $H$  is contained in  $\mathcal{L}_z$ .*

**Proof:** Recall that reparametrizations are not important: we may therefore assume that  $t_0$  is constant equal to  $1/2$ . Intuitively, the homotopy is constructed by pushing the loops towards  $t_0 = 1/2$ . More precisely, if  $\gamma = f(p)$ ,  $p \in K$ , let

$$H_1(s, p)(t) = \begin{cases} \mathfrak{F}_\gamma\left(\frac{2t}{2-s}\right) \nu_{2n}(t), & 0 \leq t \leq \frac{2-s}{4}, \\ \mathfrak{F}_\gamma\left(\frac{1}{2}\right) \nu_{2n}(t), & \frac{2-s}{4} \leq t \leq \frac{2+s}{4}, \\ \mathfrak{F}_\gamma\left(\frac{2t-s}{2-s}\right) \nu_{2n}(t), & \frac{2+s}{4} \leq t \leq 1, \end{cases} \quad s \in [0, 1],$$

and

$$H_2(s, k)(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{s}{4}, \\ \mathfrak{F}_\gamma(2t) \nu_{2n}\left(\frac{4t-s}{4-2s}\right), & \frac{s}{4} \leq t \leq \frac{1}{4}, \\ \mathfrak{F}_\gamma(1/2) \nu_{2n}\left(\frac{4t-s}{4-2s}\right), & \frac{1}{4} \leq t \leq \frac{3}{4}, \\ \mathfrak{F}_\gamma(2t-1) \nu_{2n}\left(\frac{4t-s}{4-2s}\right), & \frac{3}{4} \leq t \leq \frac{4-s}{4}, \\ \gamma(2t-1), & \frac{4-s}{4} \leq t \leq 1, \end{cases} \quad s \in [0, 1];$$

apply these two homotopies one after the other. Straightforward computations show that these curves are indeed immersions, completing the proof of the first claim. For the second claim, we need to verify that the above curves are locally convex provided  $\gamma$  is locally convex and  $n$  is sufficiently large: this is done as in the proof of Lemma 5.3 above.  $\blacksquare$

**Lemma 5.5** *Let  $K$  be a compact set,  $f : K \rightarrow \mathcal{I}_z$ . Then, for sufficiently large  $n$ , the image of  $F_{2n} \circ f$  is contained in  $\mathcal{L}_z$  and there exists  $H : [0, 1] \times K \rightarrow \mathcal{I}_z$  such that  $H(0, \cdot) = f$  and  $H(1, \cdot) = F_{2n} \circ f$ .*

Notice that the first claim (that the image is contained in  $\mathcal{L}_z$ ) is Lemma 5.3. Notice also that Proposition 1.1 follows directly from this lemma.

**Proof:** We know from Lemma 5.1 that  $f$  is homotopic to  $f^{[(1/2) \# (2n)]}$  and from Lemma 5.4 that  $f^{[(1/2) \# (2n)]}$  is homotopic to  $F_{2n} \circ f$ .  $\blacksquare$

As we shall see later, a function  $f : K \rightarrow \mathcal{L}_z \subset \mathcal{I}_z$  may be homotopic to a constant in  $\mathcal{I}_z$  but not in  $\mathcal{L}_z$ . The following proposition shows that this changes if we add loops.

**Proposition 5.6** *Let  $n$  be an even positive integer. Let  $K$  be a compact set and let  $f : K \rightarrow \mathcal{L}_z \subset \mathcal{I}_z$  a continuous function. Then  $f$  is homotopic to a constant in  $\mathcal{I}_z$  if and only if  $f^{[t_0 \# n]}$  is homotopic to a constant in  $\mathcal{L}_z$ .*

**Proof:** In  $\mathcal{I}_z$ ,  $f$  and  $f^{[t_0 \# n]}$  are homotopic, proving one implication. For the other implication, let  $H : K \times [0, 1] \rightarrow \mathcal{I}_z$  be a homotopy with  $H(\cdot, 0) = f$ ,  $H(\cdot, 1)$  constant. By Lemma 5.3, for sufficiently large even  $n$ , the image of  $F_n \circ H$  is contained in  $\mathcal{L}_z$ . This implies that  $F_n \circ f$  is homotopic in  $\mathcal{L}_z$  to a constant. From Lemma 5.5,  $f^{[t_0 \# n]}$  is homotopic to  $F_n \circ f$  in  $\mathcal{L}_I$  and therefore the proposition is proved for large  $n$ . The general case now follows from Lemma 5.2.  $\blacksquare$

A map  $f : K \rightarrow \mathcal{L}_Q$  is *loose* if  $f$  is homotopic to  $f^{[t_0 \# 2]}$  and *tight* otherwise. Lemma 5.2 shows that  $f^{[t_0 \# 2]}$  is loose. If  $K = \{p_0\}$  consists of a single point then a function  $f : K \rightarrow \mathcal{L}_Q$  is essentially a curve  $\gamma_0 = f(p_0)$ ;  $f$  is then tight if and only if  $\gamma_0$  is convex (for  $Q = I$  this follows from the results of Little; otherwise from Anisov, Shapiro and Shapiro; [15], [1], [23], [24]).

From now on we consider that a main question is, given  $f : K \rightarrow \mathcal{L}_Q$ , to decide whether  $f$  is loose or tight. We shall see a few key examples of tight maps and we shall prove that large classes of maps are loose. It is sometimes important to have estimates of the total curvature during the homotopy.

**Lemma 5.7** *Let  $f_0, f_1 : K \rightarrow \mathcal{L}_Q$  be homotopic with  $f_0$  loose. Then  $f_1$  is loose.*

**Proof:** Let  $t_0 : K \rightarrow (0, 1)$  be a continuous function and let  $H : [0, 1] \times K \rightarrow \mathcal{L}_Q$  be a homotopy from  $f_0$  to  $f_1$ . Let  $H^{[t_0 \# 2]}$  be defined by  $H^{[t_0 \# 2]}(s, p) = (H(s, p))^{[t_0 \# 2]}$ ; clearly, this is a homotopy from  $f_0^{[t_0 \# 2]}$  to  $f_1^{[t_0 \# 2]}$ . Thus, if  $f_0$  is homotopic to  $f_0^{[t_0 \# 2]}$  then  $f_1$  is homotopic to  $f_1^{[t_0 \# 2]}$ , as desired.  $\blacksquare$

We finish this section with a more complicated lemma which allows us to see that many maps  $f : K \rightarrow \mathcal{L}_Q$  are loose.

**Lemma 5.8** *Let  $Q \in SO_3$ . Let  $K$  be a compact manifold and  $f : K \rightarrow \mathcal{L}_Q$  a continuous map. Assume that  $t_0 \in (0, 1)$  and that  $t_1, t_2, \dots, t_J : K \rightarrow (0, 1)$  are continuous functions with  $t_0 < t_1 < t_2 < \dots < t_J$ . Assume that  $K = \bigcup_{1 \leq j \leq J} U_j$  where  $U_j \subset K$  are open sets. Assume that there exist continuous functions  $g_j : U_j \rightarrow \mathcal{L}_Q$  such that  $f(p) = (g_j(p))^{[t_j(p) \# 2]}$  for all  $p \in U_j$ . Then  $f$  is loose, i.e., there exists  $H : [0, 1] \times K \rightarrow \mathcal{L}_Q$  with  $H(0, p) = f(p)$ ,  $H(1, p) = (f(p))^{[t_0 \# 2]}$ . We may furthermore assume that  $\text{tot}(f(p)) \leq \text{tot}(H(s, p)) \leq 4\pi + \text{tot}(f(p))$ .*

**Proof:** Our proof proceeds by induction on  $J$ . For  $J = 1$  we have  $U_1 = K$  and therefore  $f = g_1^{[t_1 \# 2]}$ , which is known from Lemma 5.2 to be loose. The estimate on the total curvature is also given in the lemma; notice that sliding a loop between  $t_0$  and  $t_1$  does not affect total curvature.

Assume now that  $J > 1$ . Let  $W \subset U_J$  be an open set whose closure is contained in  $U_J$  and such that  $K = W \cup \bigcup_{1 \leq j \leq J-1} U_j$ . We now slide the loop in  $t_J$  to position  $t_{J-1}$  in  $W$ , allowing for the loop to stop elsewhere for  $p \in U_J \setminus W$ .



More precisely, let  $u : K \rightarrow [0, 1]$  be a continuous function with  $u(p) = 1$  for  $p \in W$  and  $u(p) = 0$  for  $p \notin U_J$ . Define  $H_J : [0, 1] \times K \rightarrow \mathcal{L}_Q$  by

$$H_J(s, p) = \begin{cases} f(p), & p \notin U_J, \\ g(p)^{[(1-u(p)s)t_J(p)+u(p)st_{J-1}(p)]\#2}, & p \in U_J. \end{cases}$$

Notice that  $\text{tot}(H_J(s, p)) = \text{tot}(f(p))$ . Let  $\hat{f}(p) = H_J(1, p)$ ,  $\hat{U}_j = U_j$  for  $j < J-1$  and  $\hat{U}_{J-1} = U_{J-1} \cup W$ ; the hypotheses of the Lemma apply to  $\hat{f}$  with a smaller value of  $J$  and therefore  $\hat{f}$  is loose. By Lemma 5.7, so is  $f$ . The estimate on total curvature also follows.  $\blacksquare$

## 6 Projective transformations and Bruhat cells

In this section we present some more algebraic notation, especially the decomposition of  $SO_3$  and  $\mathbb{S}^3$  in (signed) *Bruhat cells*. Some of this material is presented in [20] in a more algebraic fashion and for arbitrary dimension.

The *projective transformation*  $\pi(A) : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  associated to  $A \in SL_3$  is defined by  $\pi(A)(v) = \widehat{Av} = (1/|Av|)Av$ . Similarly, define  $\pi(A) : SO_3 \rightarrow SO_3$  by  $\pi(A)(Q_0) = Q_1$  if there exists  $U_1 \in Up_3^+$  with  $AQ_0 = Q_1U_1$ ; here  $Up_3^+$  is the contractible group of upper triangular matrices with positive diagonal and determinant  $+1$ . In other words,  $\pi(A)(Q_0)$  is obtained from  $AQ_0$  by performing Gram-Schmidt on columns. In particular, if  $A \in SO_3$  then  $\pi(A)(Q_0) = AQ_0$ . This action of  $SL_3$  on  $SO_3$  is transitive but not doubly transitive; we shall soon discuss the extent to which it fails to be doubly transitive.

Notice that  $\pi(A) : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is an orientation preserving diffeomorphism. For an immersion  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$  define  $\pi(A)\gamma$  by  $(\pi(A)\gamma)(t) = \pi(A)(\gamma(t))$ ; the curve  $\pi(A)\gamma$  is again an immersion. Since  $\pi(A)$  takes geodesics to geodesics, if  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$  is locally convex then so is  $\pi(A)\gamma$ ; (this can also be checked directly from the definition by a straightforward computation). The map  $\pi(A)$  is defined so that  $\mathfrak{F}_{\pi(A)\gamma} = \pi(A)(\mathfrak{F}_\gamma)$  for any immersion  $\gamma : [0, 1] \rightarrow \mathbb{S}^2$ . Notice that  $\pi(A)I = I$  if and only if  $A \in Up_3^+$ . Thus, for  $U \in Up_3^+$ ,  $\pi(U)$  defines a smooth map from  $\mathcal{L}_Q$  to  $\mathcal{L}_{\pi(U)Q}$ .

Let  $B_3 \subset O_3$  be the Coxeter-Weyl group of signed permutation matrices: the group  $B_3$  has 48 elements and corresponds to the isometries of the octahedron of vertices  $\pm e_1, \pm e_2, \pm e_3$ . Let  $B_3^+ = B_3 \cap SO_3$  be the subgroup of orientation preserving isometries. Dropping signs defines a homomorphism from  $B_3^+$  to the symmetric group  $S_3$ : the *number of inversions* of  $Q \in B_3^+$  is the number of inversions of the corresponding permutation in  $S_3$ . We denote an element of  $B_3^+$

by the letter  $P$  (for permutation), indicating in the subscript the corresponding permutation (as a cycle) and the signs, read by column, in binary:

$$0 = + + +, 1 = + + -, 2 = + - +, 3 = + - -, \dots, 7 = - - -;$$

thus, for instance

$$P_{(13);1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & +1 & 0 \\ +1 & 0 & 0 \end{pmatrix}, \quad P_{(13);2} = \begin{pmatrix} 0 & 0 & +1 \\ 0 & -1 & 0 \\ +1 & 0 & 0 \end{pmatrix}.$$

The lift  $\tilde{B}_3^+ \subset \mathbb{S}^3$  of  $B_3^+$  is the group of 48 quaternions with either one coordinate of absolute value 1 and three equal to 0 or two coordinates of absolute value  $1/\sqrt{2}$  and two equal to 0 or four coordinates of absolute value  $1/2$ . For instance,

$$\Pi\left(\frac{\mathbf{1} - \mathbf{j}}{\sqrt{2}}\right) = P_{(13);1}, \quad \Pi\left(\frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}\right) = P_{(13);2}.$$

The Bruhat cell  $\text{Bru}(Q) \subset SO_3$  of  $Q \in SO_3$  is the set of all orthogonal matrices of the form  $U_0 Q U_1^{-1}$ , with  $U_0, U_1 \in Up_3^+$ . Each Bruhat cell contains a unique element of  $B_3^+$ . We also denote Bruhat by  $\text{Bru}_*$ , with subscripts defined as for  $P_* \in B_3^+$ ; thus, for instance,  $\text{Bru}(P_{(13);1}) = \text{Bru}_{(13);1}$ . The cell  $\text{Bru}(P)$  ( $P \in B_3^+$ ) is diffeomorphic to  $\mathbb{R}^d$ , where  $d$  is the number of inversions of  $P$ . There are therefore exactly four open cells, corresponding to the two matrices above plus

$$\Pi\left(\frac{\mathbf{1} + \mathbf{j}}{\sqrt{2}}\right) = P_{(13);4}, \quad \Pi\left(\frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}\right) = P_{(13);7}.$$

If  $Q_0$  and  $Q_1$  belong to the same Bruhat cell then there exists a canonical and explicit diffeomorphism between the Hilbert manifolds  $\mathcal{L}_{Q_0}$  and  $\mathcal{L}_{Q_1}$ . Indeed, for  $Q_0$  and  $Q_1 \in \text{Bru}(Q_0)$  let  $U_0, U_1 \in Up_3^+$  be such that  $U_0 Q_0 = Q_1 U_1$ ;  $U_0$  is uniquely defined if we require  $U_0 \in Up_3^1$ , where  $Up_3^1 \subset Up_3^+$  is the subgroup of matrices with diagonal entries equal to  $+1$ . Then the map  $\pi(U_0) : \mathcal{L}_{Q_0} \rightarrow \mathcal{L}_{Q_1}$  is a diffeomorphism; its inverse is the similarly constructed map  $\pi(U_0^{-1})$ .

Figure 6 shows example of curves  $\gamma \in \mathcal{L}_Q$  for  $Q \in \text{Bru}_{(13);\ell}$  for  $\ell = 1, 2, 4, 7$  (in this order).

The dashed closed convex curves indicate a convenient geometric way to recognize these Bruhat cells: in all cases there exists a closed convex curve tangent to both endpoints of  $\gamma$  and orientations at the endpoints allow us to distinguish between cells.

For  $\gamma \in \mathcal{L}$ , let  $\Gamma = \mathfrak{F}_\gamma : [0, 1] \rightarrow SO_3$  and write  $\mathfrak{F}_\gamma(t_0; t) = \Gamma(t_0; t) = (\Gamma(t_0))^{-1} \Gamma(t)$ . Similarly,  $\tilde{\mathfrak{F}}_\gamma(t_0; t) = \tilde{\Gamma}(t_0; t) = (\tilde{\Gamma}(t_0))^{-1} \tilde{\Gamma}(t)$  where  $\tilde{\Gamma} : [0, 1] \rightarrow \mathbb{S}^3$  is a lift of  $\Gamma$ . Clearly,  $\Gamma(0; t) = \Gamma(t)$ ,  $\Gamma(t_0; t_0) = I$ ,  $\tilde{\Gamma}(t_0; t_0) = \mathbf{1}$ . On the other hand, there exists  $\epsilon > 0$  such that for all  $t \in (t_0, t_0 + \epsilon)$  we have  $\Gamma(t_0; t) \in \text{Bru}_{(13);2}$ .

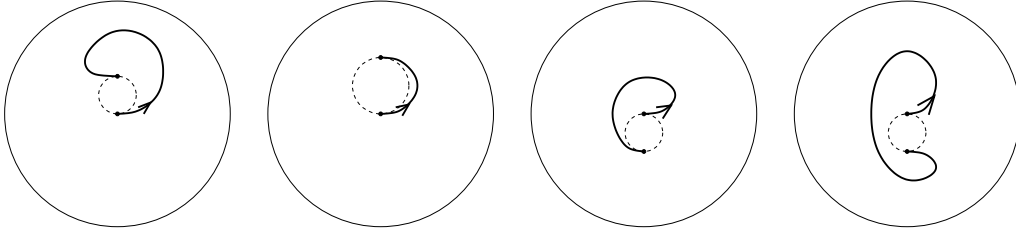


Figure 6: Representatives of the open Bruhat cells

A locally convex arc  $\gamma|_{[t_0, t_1]}$  is *convex* if  $\mathfrak{F}_\gamma(t_0; t) \in \text{Bru}_{(13);2}$  for all  $t \in (t_0, t_1)$ . Notice that we do not require that  $\mathfrak{F}_\gamma(t_0; t_1) \in \text{Bru}_{(13);2}$ . Indeed, there are five other Bruhat cells to which  $\mathfrak{F}_\gamma(t_0; t_1)$  may belong:  $\text{Bru}_{(123);6}$ ,  $\text{Bru}_{(132);0}$ ,  $\text{Bru}_{(23);2}$ ,  $\text{Bru}_{(12);4}$  and  $\text{Bru}_{e;0} = I$ . Examples of convex arcs corresponding to these five cells are given in Figure 7. We shall come back to these five cells again and again.

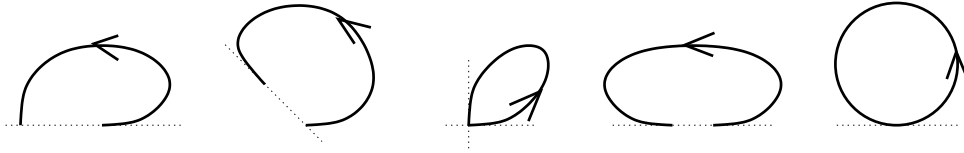


Figure 7: Convex arcs

Recall that a matrix  $Q \in SO_3$  is *convex* if there exists a convex arc  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$  with  $\mathfrak{F}_\gamma(t_0; t_1) = Q$ ; thus, the set of convex matrices is the union of the six Bruhat cells above (including the open cell  $P_{(13);2}$ ). Also, a quaternion  $z \in \mathbb{S}^3$  is *convex* if there exists a convex arc  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$  with  $\tilde{\mathfrak{F}}_\gamma(t_0; t_1) = z$ ; the six convex elements of  $\tilde{B}_3^+$  are

$$\frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \quad \frac{-\mathbf{1} + \mathbf{i} - \mathbf{j} + \mathbf{k}}{2}, \quad \frac{-\mathbf{1} + \mathbf{i} + \mathbf{j} + \mathbf{k}}{2}, \quad \frac{-\mathbf{1} + \mathbf{i}}{\sqrt{2}}, \quad \frac{-\mathbf{1} + \mathbf{k}}{\sqrt{2}}, \quad -\mathbf{1}.$$

A quaternion  $z$  is *anticonvex* if  $-z$  is convex. The sets of convex and anticonvex quaternions are unions of distinct Bruhat cells and therefore distinct; furthermore, the only points of intersection between their respective closures are  $\pm \mathbf{1}$ .

## 7 Multiconvex curves

A curve  $\gamma \in \mathcal{L}_z$  is *multiconvex* of multiplicity  $k$  if there exist  $0 = t_0 < t_1 < \dots < t_k = 1$  such that

- (a)  $\mathfrak{F}_\gamma(t_i) = I$  for  $i < k$ ;

(b) the restrictions  $\gamma|_{[t_{i-1}, t_i]}$  are convex arcs for  $i \leq k$ .

Notice that for  $i < k$  these restrictions are then simple closed curves (see Figure 3). Let  $\mathcal{M}_k \subset \mathcal{L}_z$  be the set of multiconvex curves of multiplicity  $k$ . A curve  $\gamma \in \mathcal{L}_z$  is multiconvex of multiplicity 1 if and only if it is convex, so that for  $z = -1$  we have  $\mathcal{M}_1 = \mathcal{L}_{-1, c}$ . By Lemma 3.1, if  $\gamma \in \mathcal{M}_k$  then  $2(k-1)\pi < \text{tot}(\gamma) < 4k\pi$ . It is easy to see that, for  $k$  odd,  $\mathcal{M}_k \neq \emptyset$  if and only if  $z$  is convex; similarly, for  $k$  even,  $\mathcal{M}_k \neq \emptyset$  if and only if  $-z$  is convex.

**Lemma 7.1** *Let  $z \in \mathbb{S}^3$ . Let  $k$  be a positive integer. The closed subset  $\mathcal{M}_k \subset \mathcal{L}_z$  (if non-empty) is a contractible submanifold of codimension  $2k - 2$  with trivial normal bundle.*

**Proof:** Assume that  $-z$  is convex. Consider the geodesic  $\rho \subset \mathbb{S}^2$  passing through  $\pm e_1$  and  $\pm e_3$  (where  $e_1 = (1, 0, 0)$ ). After a projective transformation we may assume that any convex curve  $\gamma \in \mathcal{L}_{-z}$  crosses  $\rho$  transversally once for some  $t \in (0, 1)$ .

We first define open sets  $U_k \subset \mathcal{L}_{(-1)^k z}$ . A curve  $\gamma \in \mathcal{L}_{(-1)^k z}$  belongs to  $U_k$  if and only if:

- (a) all intersections between  $\gamma$  and  $\rho$  are transversal;
- (b) there are exactly  $2k$  values

$$0 = t_0 < t_{\frac{1}{2}} < t_1 < \cdots < t_{k-1} < t_{k-\frac{1}{2}} < 1$$

of  $t \in [0, 1)$  for which  $\gamma(t) \in \rho$ ;

- (c) consecutive intersections  $\gamma(t_j)$  and  $\gamma(t_{j \pm \frac{1}{2}})$  are distinct;
- (d) arcs of  $\gamma$  between consecutive intersections are convex.

Notice that  $U_k$  is indeed open and  $t_j : U_k \rightarrow [0, 1]$  are continuous functions. We may continuously define the arguments  $\theta_j$  of the points  $\gamma(t_j(\gamma))$  by

$$\gamma(t_j(\gamma)) = \cos(\theta_j(\gamma))e_1 + \sin(\theta_j(\gamma))e_3, \quad \theta_0(\gamma) = 0,$$

and, for integer  $j$ ,

$$\theta_j < \theta_{j+\frac{1}{2}} < \theta_j + \pi, \quad \theta_{j+\frac{1}{2}} > \theta_{j+1} > \theta_{j+\frac{1}{2}} - \pi.$$

Also,  $\langle \gamma'(t_j(\gamma)), e_2 \rangle$  is positive if  $j$  is an integer (and negative otherwise). For  $j$  an integer, we continuously define the argument  $\eta_j$  of  $\gamma'(t_j(\gamma))$  by

$$\begin{aligned} \gamma'(t_j(\gamma)) &= r (\cos(\eta_j(\gamma))e_2 + \sin(\eta_j(\gamma))n), \\ n &= -\sin(\theta_j(\gamma))e_1 + \cos(\theta_j(\gamma))e_3, \quad -\pi/2 < \eta_j(\gamma) < \pi/2, \quad r > 0. \end{aligned}$$

Now define  $M_k : U_k \rightarrow \mathbb{R}^{2k-2}$  by

$$M_k(\gamma) = (\theta_1(\gamma), \eta_1(\gamma), \dots, \theta_{k-1}(\gamma), \eta_{k-1}(\gamma)).$$

The smooth map  $M_k$  is a submersion and  $\mathcal{M}_k$  is the inverse image of  $0 \in \mathbb{R}^{2k-2}$ . This proves that  $\mathcal{M}_k$  is a smooth submanifold of codimension  $2k-2$  and trivializes its normal bundle.

Finally, we must prove that  $\mathcal{M}_k$  is contractible. The fact that  $\mathcal{L}_{-z,c} = \mathcal{M}_1(z)$  is contractible is well known ([1] and [24], Lemma 5). The subset  $\hat{\mathcal{M}}_k$  of  $\mathcal{M}_k$  of curves for which  $t_j = j/k$  is naturally identified with  $(\mathcal{M}_1(-1))^{(k-1)} \times (\mathcal{M}_1(z))$  (reparametrize  $\gamma|_{[t_j, t_{j+1}]}$  to define  $\gamma_j \in \mathcal{M}_1$ ) and therefore is also contractible. But  $\hat{\mathcal{M}}_k \subset \mathcal{M}_k$  is a deformation retract: just use piecewise affine functions to reparametrize each curve so that  $t_j = j/k$  (for all  $j$ ). ■

The previous result allows us to use each  $\mathcal{M}_k$  to define an element  $m_{2k-2} \in H^{2k-2}(\mathcal{L}_z)$  by counting intersections with  $\mathcal{M}_k$ . For de Rham cohomology, for instance, we consider Thom's form in the (trivial) normal bundle to  $\mathcal{M}_k$  and use the identification of this bundle with a tubular neighborhood of  $\mathcal{M}_k$  to define a closed  $(2k-2)$ -form  $\omega$  which is a representative of  $m_{2k-2}$ . Thus, if  $f : K \rightarrow \mathcal{L}_z$  is a smooth map from an oriented compact  $(2k-2)$ -dimensional manifold  $K$  to  $\mathcal{L}_z$  which is transversal to  $\mathcal{M}_k$  then the integral of the pull back of  $\omega$  over  $K$  equals the number of intersections of  $f$  with  $\mathcal{M}_k$ , counted with sign. If the map is not smooth or not (topologically) transversal we may perturb it so that it becomes both smooth and transversal: the number of intersections is still well defined. We denote this integer by  $m_{2k-2}(f)$ .

In the following lemma the sphere  $\mathbb{S}^2$  will be the compact manifold (usually called  $K$ ) in the domain of a map. Let  $\mathbf{s} = -e_3$ ,  $\mathbf{n} = e_3$  be the south and north pole, respectively. The base point of  $\mathbb{S}^2$  is  $\mathbf{s}$ .

**Lemma 7.2** *There exist maps  $g_s : \mathbb{S}^2 \rightarrow \mathcal{L}_1$ ,  $s \in [0, \frac{1}{2})$ , such that:*

- (a)  $g_s(\mathbf{s}) = \nu_2$  and  $g_s(\mathbf{n})$  is a reparametrization of  $\nu_4$ ;
- (b) if  $g_s(p)$  is multiconvex then  $p = \mathbf{s}$  or  $p = \mathbf{n}$ ;
- (c)  $g_s$  is topologically transversal to  $\mathcal{M}_2$  at  $\mathbf{s}$ ;
- (d) if  $t \in [1-s, 1]$  then  $(g_s(p))(t) = \nu_2(t)$ ;
- (e) if  $\tilde{\mathfrak{F}}_{g_s(p)}(t) \in \mathcal{C}_{\mathbf{k}}$ ,  $t \in (0, 1)$ , then either  $p = \mathbf{s}$ ,  $p = \mathbf{n}$  or  $t \in [1-s, 1]$ ;
- (f) given  $t \in (0, 1-s)$ , the map  $p \mapsto \tilde{\mathfrak{F}}_{g_s(p)}(t)$  is topologically transversal to  $\mathcal{C}_{\mathbf{k}}$ ;
- (g) the maps  $g_s$  are all homotopic to  $g_0$ ;
- (h) the maps  $g_s$  satisfy  $m_2(g_s) = \pm 1$  and are all tight.

The sign ambiguity in the last item comes from the fact that we were not too careful to define either a standard transversal orientation to  $\mathcal{M}_2$  or a standard orientation for  $K = \mathbb{S}^2$ , the domain of  $g_s$ . Recall that  $\mathcal{C}_{\mathbf{k}} \subset \mathbb{S}^3$  is a subgroup defined before Lemma 4.2.

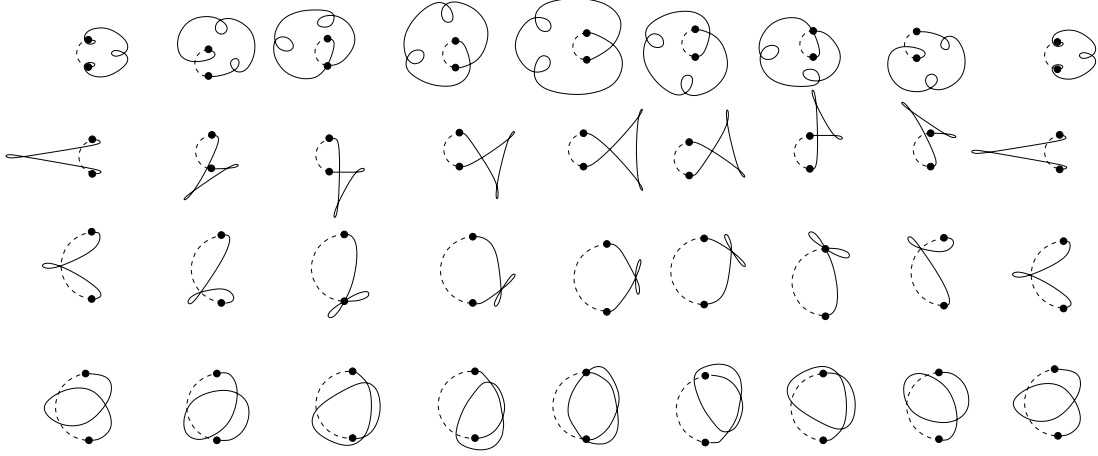


Figure 8: A sketch of  $g_s : \mathbb{S}^2 \rightarrow \mathcal{L}_1$ .

Figure 8 gives a sketch of  $g_s$  for  $s = 1/4$ . Here the north and south poles are outside the figure, to the top and bottom, respectively. Meridians ( $\theta$  constant) are vertical and parallels ( $\alpha$  constant) are horizontal (as in most world maps). The curves go around the sphere between lines 1 and 2.

The proof of the lemma will consist of an explicit construction of a map  $g_s$ . It should be noted that many arbitrary choices are made during the construction, sometimes in order to facilitate some later argument.

Before turning to the proof of this lemma and the construction of the maps  $g_s$  we need a few remarks. A circle of radius  $\rho < \pi/2$  is a closed convex curve:

$$\gamma(t) = \cos(2\pi t) \sin(\rho) v_1 + \sin(2\pi t) \sin(\rho) v_2 + \cos(\rho) v_3, \quad t \in [0, 1],$$

where  $v_1, v_2, v_3$  is a positively oriented orthonormal basis. The image of a convex circle by a projective transformation is a *spherical ellipse*, or just *ellipse*. Notice that for us an ellipse is an oriented curve. Also, a *good parametrization* of an ellipse is a locally convex curve  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$  of the form  $\gamma = \pi(A) \circ \tilde{\gamma}$  where  $A \in SL_3$  and  $\tilde{\gamma}$  is a parametrization by a multiple of arc length of a circle.

Recall that two smooth curves *osculate* each other at a common point if that point is a point of tangency of order three. Given  $Q \in \text{Bru}_{(13);2}$  there exists a unique ellipse  $\mathcal{E}$  and good parametrization  $\gamma : [0, 1] \rightarrow \mathcal{E} \subset \mathbb{S}^2$  with  $\gamma$  convex,  $\gamma(0) = e_1$ ,  $\gamma'(0) = e_2$ ,  $\mathcal{E}$  osculating the circle  $\mathcal{C}_{\mathbf{k}}$  at  $e_1$  and  $\mathfrak{F}_{\gamma}(1) = Q$ .

**Proof:** We first construct  $g_0$  (explicitly). Let  $\tilde{\Gamma}_\alpha$  be as in Section 4, shortly before Lemma 4.1. Notice that

$$\tilde{\Gamma}_\alpha \left( t + \frac{1}{3} \right) = \exp \left( \frac{\pi}{3} \mathbf{k} \right) \tilde{\Gamma}_\alpha(t); \quad \tilde{\Gamma}_\alpha \left( t + \frac{1}{2} \right) = -\tilde{\Gamma}_{-\alpha}(t).$$

Define  $g_0 : \mathbb{S}^2 \rightarrow \mathcal{L}_1$  by

$$g_0(p)(t) = \left( \Gamma_\alpha \left( \frac{\theta}{6\pi} \right) \right)^{-1} \gamma_\alpha \left( t + \frac{\theta}{6\pi} \right), \quad p = (\cos \theta \sin \alpha, \sin \theta \sin \alpha, -\cos \alpha).$$

Notice that  $g_0$  is well defined. We prove that  $g_0$  satisfies the desired properties.

Items (a) and (d) are immediate and item (e) follows directly from Lemma 4.2. Item (b) is a direct consequence of item (e). For items (c) and (f) we first notice that we are talking about isolated points. Indeed, from (b),  $p = \mathbf{s}$  is the only point for which  $g_0(p) \in \mathcal{M}_2$ . Similarly, from (e),  $p = \mathbf{s}$  is the only point for which  $\mathfrak{F}_{g_0(p)}(t) \in \mathcal{C}_{\mathbf{k}}$ . In either case we need to study what happens when  $p$  goes around  $\mathbf{s}$ , drawing a small circle. For (c), we need to prove that  $g_0(p)$  will go around  $\mathcal{M}_2$  once, or, equivalently, that  $M_2(g_0(p))$  will go around the origin once. For (f), we need to prove that  $\mathfrak{F}_{g_0(p)}(t)$  will go around the circle  $\mathcal{C}_{\mathbf{k}}$  once. Unfortunately,  $g_0$  is not differentiable at  $\mathbf{s}$  but it does admit directional derivatives: that is enough. We go back to the construction of  $\tilde{B}_\alpha$  and  $\tilde{\Gamma}_\alpha$  in order to compute directional derivatives of  $g_0$ . For  $t \in [-1/12, 1/12]$ , we may translate the geometric description in Section 4 as:

$$\begin{aligned} \tilde{B}_\alpha(t) &= \exp \left( \frac{\alpha}{2} \mathbf{j} \right) \exp(u(\alpha)t\mathbf{k}) \exp(v(\alpha)\mathbf{j}), \\ u(\alpha) &= 6 \arccos \left( \frac{\cos \alpha}{\sqrt{4 - \sin^2 \alpha}} \right), \quad v(\alpha) = -\frac{1}{2} \arcsin \left( \frac{1}{2} \sin \alpha \right). \end{aligned}$$

Notice that  $u'(0) = 0$ ,  $v'(0) = -1/4$ . Let  $\tilde{B}_\alpha^\bullet(t)$  be the derivative of  $\tilde{B}_\alpha(t)$  with respect to  $\alpha$ ; we have  $(\tilde{B}_0(t))^{-1} \tilde{B}_0^\bullet(t) = w(t)$  where the auxiliary function  $w$  is defined by

$$w(t) = \frac{1}{4} ((2 \cos(4\pi t) - 1) \mathbf{j} + (-2 \sin(4\pi t)) \mathbf{i}).$$

It is now easy to obtain similar formulas for other intervals and to deduce that

$$(\tilde{B}_0(t))^{-1} \tilde{B}_0^\bullet(t) = \begin{cases} w(t - t_0), & t \in [t_0 - \frac{1}{12}, t_0 + \frac{1}{12}], t_0 = \frac{k}{3}, \\ -w(t - t_0), & t \in [t_0 - \frac{1}{12}, t_0 + \frac{1}{12}], t_0 = \frac{k}{3} + \frac{1}{6}, \end{cases} \quad k \in \mathbb{Z}.$$

Thus  $(\tilde{B}_0(t))^{-1} \tilde{B}_0^\bullet(t)$ , as a function of  $t$ , performs three full turns around the origin in the plane spanned by  $\mathbf{i}$  and  $\mathbf{j}$ . We now have that  $(B_0(t; t + \frac{1}{2}))^{-1} B_0^\bullet(t; t + \frac{1}{2})$  performs one full turn around the origin when  $t$  goes from 0 to  $1/3$ .

Let  $p = (\cos \theta \sin \alpha, \sin \theta \sin \alpha, -\cos \alpha)$ ; let  $M_2$  and  $U_2$  be as in Lemma 7.1. For sufficiently small  $\alpha$ , we have  $g_0(p) \in U_2$ . From the above computations, for sufficiently small  $\alpha$ ,  $M_2(g_0(p))$  also performs a full turn around the origin when  $\theta$  goes from 0 to  $2\pi$ , completing the proof of (c). Item (f) follows similarly from these computations for  $t = 1/2$ ; we notice that, by continuity, the number of turns of  $\tilde{\mathfrak{F}}_{g_0(p)}(t)$  around  $\mathcal{C}_{\mathbf{k}}$  must be the constant; this completes the proof of item (f).

Notice now that items (a) through (f) imply that  $g_0$  intersects  $\mathcal{M}_2$  topologically transversally and exactly once. Thus, for  $m_2 \in H^2$  as above,  $m_2(g_0) = \pm 1$ . On the other hand,  $g_0^{[t_0; 2]}$  can not possibly intersect  $\mathcal{M}_2$  and therefore  $m_2(g_0^{[t_0; 2]}) = 0 \neq m_2(g_0)$ . Thus  $g_0$  and  $g_0^{[t_0; 2]}$  are not homotopic and  $g_0$  is therefore tight. This completes the proof of item (h) and of the case  $s = 0$ .

We now construct  $g_s$  for  $s > 0$ ,  $s$  small. By compactness, there exists  $\epsilon_1 > 0$  such that for all  $p \in \mathbb{S}^2$ , the arc  $g_0(p)|_{[-2\epsilon_1, +2\epsilon_1]}$  is convex. Here we interpret  $g_0(p)$  as a 1-periodic function from  $\mathbb{R}$  to  $\mathbb{S}^2$ . Recall that  $\tilde{\mathfrak{F}}_{g_0(p)}(0) = \tilde{\mathfrak{F}}_{\nu_2}(0) = \mathbf{1}$ . Again by compactness, there exists  $\epsilon_2 \in (0, \epsilon_1/4)$  such that, for all  $t \in [0, \epsilon_2]$  and for all  $p \in \mathbb{S}^2$ , we have

$$(\tilde{\mathfrak{F}}_{\nu_2}(t))^{-1} \tilde{\mathfrak{F}}_{g_0(p)}(\epsilon_1), (\tilde{\mathfrak{F}}_{g_0(p)}(-\epsilon_1))^{-1} \tilde{\mathfrak{F}}_{\nu_2}(-t) \in \text{Bru}_{(13); 2}.$$

We want to define  $\hat{g} : \mathbb{S}^2 \rightarrow \mathcal{L}_1$  with

$$(\hat{g}(p))(t) = \begin{cases} \nu_2(t), & t \in [0, \epsilon_2] \cup [1 - \epsilon_2, 1], \\ (g_0(p))(t), & t \in [\epsilon_1, 1 - \epsilon_1]. \end{cases}$$

As in the remark above, for each  $p \in \mathbb{S}^2$ , there exist ellipses  $\mathcal{E}_+$  and  $\mathcal{E}_-$  and good parametrizations  $\gamma_+ : [\epsilon_2, \epsilon_1] \rightarrow \mathcal{E}_+ \subset \mathbb{S}^2$  and  $\gamma_- : [1 - \epsilon_1, 1 - \epsilon_2] \rightarrow \mathcal{E}_- \subset \mathbb{S}^2$  such that

$$\begin{aligned} \gamma_+(\epsilon_2) &= \nu_2(\epsilon_2), \quad \gamma'_+(\epsilon_2) = \nu'_2(\epsilon_2), \quad \tilde{\mathfrak{F}}_{\gamma_+}(\epsilon_1) = \tilde{\mathfrak{F}}_{g(p)}(\epsilon_1), \\ \gamma_-(1 - \epsilon_2) &= \nu_2(1 - \epsilon_2), \quad \gamma'_-(1 - \epsilon_2) = \nu'_2(1 - \epsilon_2), \quad \tilde{\mathfrak{F}}_{\gamma_-}(1 - \epsilon_1) = \tilde{\mathfrak{F}}_{g(p)}(1 - \epsilon_1), \end{aligned}$$

and, furthermore,  $\mathcal{E}_+$  and  $\mathcal{E}_-$  osculate the circle  $\mathcal{C}_{\mathbf{k}}$  at  $\nu_2(\epsilon_2)$  and  $\nu_2(1 - \epsilon_2)$ , respectively. The ellipses and parametrizations are uniquely and continuously defined. Complete the definition of  $\hat{g}$  by

$$(\hat{g}(p))(t) = \begin{cases} \gamma_+(t), & t \in [\epsilon_2, \epsilon_1], \\ \gamma_-(t), & t \in [1 - \epsilon_1, 1 - \epsilon_2]. \end{cases}$$

Notice that since 5 points define a conic, the ellipses  $\mathcal{E}_{\pm}$  have no tangency point to  $\mathcal{C}_{\mathbf{k}}$  besides  $\nu_2(\pm\epsilon_2)$ .

For  $s \leq 2\epsilon_2$ , set  $g_s(p)(t) = (\tilde{\mathfrak{F}}_{\nu_2}(\epsilon_2))^{-1} \hat{g}(t - \epsilon_2)$ . We claim that the function  $g_s$  has all the required properties. Item (a) is obvious. Item (d) holds by construction



and item (e) follows from the last observation in the previous paragraph; item (b) now follows. Topological transversality (items (c) and (f)) is handled as for  $s = 0$ . Item (g) follows either from an explicit computation or from the contractibility of  $\mathcal{L}_{z,c}$  and item (h) now follows. This completes the proof of the theorem for  $s \leq 2\epsilon_2$ .

For  $c \in \mathbb{R}$ , let

$$A(c) = \begin{pmatrix} 1 & c & c^2/2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \exp \left( c \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Notice that  $\pi(A(c))e_1 = e_1$ ,  $\pi(A(c))I = I$  and  $\pi(A(c))$  takes the circle  $\mathcal{C}_{\mathbf{k}}$  to itself. Let  $\hat{g}_c : \mathbb{S}^2 \rightarrow \mathcal{L}_1$  be given by  $\hat{g}_c(p) = \pi(A(c)) \circ g_{\epsilon_2}(p)$ . Given  $c$ , each arc  $(\hat{g}_c(p))_{[1-\epsilon_2, 1]}$  is a fixed parametrization of an arc of  $\mathcal{C}_{\mathbf{k}}$ . Changing  $c$ , that arc can be taken to have any required length. In other words, given  $s \in (\epsilon_2, \frac{1}{2})$  there exists a unique  $c \in \mathbb{R}$  for which  $\mathfrak{F}_{\hat{g}_c(p)}(1 - \epsilon_2) = \mathfrak{F}_{\nu_2}(1 - s)$ : define  $g_s$  by suitably reparametrizing this  $\hat{g}_c$ . All the required properties follow by construction. ■

**Lemma 7.3** *Let  $k > 1$  be a positive integer. Let  $z \in \mathbb{S}^3$  with  $-z$  convex. There exist tight maps  $h_{2k-2} : \mathbb{S}^{2k-2} \rightarrow \mathcal{L}_{(-1)^k z}$  which:*

- (a) *intersect  $\mathcal{M}_k$  exactly once and topologically transversally;*
- (b) *do not intersect  $\mathcal{M}_{k'}$  for  $k' \neq k$ ;*
- (c) *satisfy  $m_{2k-2}(h_{2k-2}) = \pm 1$ ;*
- (d) *are homotopic to a constant as maps  $\mathbb{S}^{2k-2} \rightarrow \mathcal{I}_{(-1)^k z}$ .*

**Proof:** Let  $\mathbb{D}^2 \subset \mathbb{R}^2$  be the closed disk of radius 1. We first construct a function  $\hat{h} : (\mathbb{D}^2)^{(k-1)} \rightarrow \mathcal{L}_{(-1)^k z}$ . Consider  $\epsilon_0 > 0$  such that if  $t \in (0, \epsilon_0)$  then  $-(\tilde{\mathfrak{F}}_{\nu_1}(kt))^{-1}z$  is convex. Let  $s_0 = k(1 + \epsilon_0)$  and  $z_0 = \tilde{\mathfrak{F}}_{\nu_{s_0}}(\frac{1}{k}) = -\tilde{\mathfrak{F}}_{\nu_1}(\epsilon_0)$ . We shall have  $(\hat{h}(0))(t) = \nu_{s_0}(t)$  for  $t \in [0, \frac{k-1}{k}]$  and  $\tilde{\mathfrak{F}}_{\hat{h}(p)}(\frac{i}{k}) = z_0^i$  for all  $p$  and for all integers  $i < k$  (in other words, we are starting the definition here; the expression “we shall have” is to be understood as: “here is yet another property of  $\hat{h}$ , which is clearly consistent with what we demanded before”).

Let  $\gamma_k : [\frac{k-1}{k}, 1] \rightarrow \mathbb{S}^2$  be a convex arc with  $\tilde{\mathfrak{F}}_{\gamma_k}(\frac{k-1}{k}) = z_0^{k-1}$ ,  $\tilde{\mathfrak{F}}_{\gamma_k}(1) = (-1)^k z$ . We shall have  $(\hat{h}(p))(t) = \gamma_k(t)$  for all  $p \in (\mathbb{D}^2)^{(k-1)}$  and  $t \in [\frac{k-1}{k}, 1]$ . Let  $s_1 = 1 - \frac{s_0}{2k} = \frac{1-\epsilon_0}{2}$  and let  $g_{s_1}$  be as in Lemma 7.2 so that  $\tilde{\mathfrak{F}}_{g_{s_1}(p)}(1 - s_1) = z_0$  for all  $p \in \mathbb{S}^2$ . Recall that  $\tilde{\nu}_4 = g_{s_1}(\mathbf{n})$  is a reparametrization of  $\nu_4$  with  $\tilde{\nu}_4(t) = \nu_2(t)$  for all  $t > 1 - s_1$ . Define  $w : \mathbb{D}^2 \rightarrow \mathbb{S}^2$  by

$$w(r \cos \theta, r \sin \theta) = \begin{cases} (\cos \theta \sin(4r), \sin \theta \sin(4r), -\cos(4r)), & r \leq \pi/4, \\ (0, 0, 1), & r \geq \pi/4. \end{cases}$$

Consider  $p = (p_1, p_2, \dots, p_{k-1}) \in (\mathbb{D}^2)^{(k-1)}$ ,  $p_i \in \mathbb{D}^2$ . For  $t \in [\frac{i-1}{k}, \frac{i}{k}]$  let  $t_i = (1 - s_1)k(t - \frac{i-1}{k})$ ; if  $|p_i| \leq \frac{\pi}{4}$  we shall have  $(\hat{h}(p))(t) = \Pi(z_0^{i-1})g_{s_1}(w(p_i))(t_i)$ . Let  $\tilde{\nu}_{8k}$  be a reparametrization of  $\nu_{8k}$  with  $\tilde{\nu}_{8k}(t) = \nu_2(t)$  for all  $t > 1 - s_1$ . Let  $\tilde{g} : [\frac{\pi}{4}, \frac{7}{8}]$  be a path from  $\tilde{g}(\frac{\pi}{4}) = \tilde{\nu}_4$  to  $\tilde{g}(\frac{7}{8}) = \tilde{\nu}_{8k}$  satisfying  $\tilde{g}(\tau)(t) = \nu_2(t)$  for all  $t > 1 - s_1$ ; notice that  $\tilde{\mathfrak{F}}_{\tilde{g}(\tau)}(1 - s_1) = z_0$  for all  $\tau \in [\frac{\pi}{4}, \frac{7}{8}]$ . For  $t \in [\frac{i-1}{k}, \frac{i}{k}]$  and  $t_i$  as above we shall have

$$(\hat{h}(p))(t) = \begin{cases} \Pi(z_0^{i-1})\tilde{g}(|p_i|)(t_i), & |p_i| \in [\frac{\pi}{4}, \frac{7}{8}], \\ \Pi(z_0^{i-1})\tilde{\nu}_{8k}(t_i), & |p_i| \geq \frac{7}{8}, \end{cases}$$

completing the construction of  $\hat{h}$ .

From Lemma 7.2,  $\hat{h}(0) \in \mathcal{M}_k$ ,  $\hat{h}(p) \in \mathcal{M}_k$  implies  $p = 0$  and  $\hat{h}$  is topologically transversal to  $\mathcal{M}_k$ , with a single intersection at  $p = 0$ . Consider

$$\hat{h}_1 = \hat{h}|_{\partial((\mathbb{D}^2)^{(k-1)})} : \partial((\mathbb{D}^2)^{(k-1)}) \rightarrow \mathcal{L}_{(-1)^{k_z}}.$$

We apply Lemma 5.8 to prove that  $\hat{h}_1$  is loose: here  $K = \partial((\mathbb{D}^2)^{(k-1)})$  (which is homeomorphic to  $\mathbb{S}^{(2k-3)}$ ),  $J = k-1$ ,  $t_j = \frac{j}{k} - \frac{1}{2k}$  and  $(p_1, p_2, \dots, p_{k-1}) \in U_j$  if  $i \neq j$  implies  $|p_i| > \frac{7}{8}$ . There exists therefore a homotopy  $H : [0, 1] \times \partial((\mathbb{D}^2)^{(k-1)}) \rightarrow \mathcal{L}_{(-1)^{k_z}}$  with  $H(0, p) = \hat{h}_1(p)$  and  $H(1, p) = (\hat{h}_1(p))^{[t_k \# 2]}$ ,  $t_k = 1 - \frac{1}{2k}$ .

The homotopy  $H$  may be assumed to be disjoint from  $\mathcal{M}_k$ . In order to see this we give estimates on the total curvature. The total curvature of  $\tilde{\nu}_{8k}$  equals  $16k\pi$ , and the total curvature of its restriction to  $[0, 1 - s_1]$  is greater than  $(16k - 2)\pi$ . If  $p \in \partial((\mathbb{D}^2)^{(k-1)})$  we have at least one index  $j$  for which  $|p_j| = 1$ ; for such  $j$ , the total curvature in the interval  $[\frac{j-1}{k}, \frac{j}{k}]$  is greater than  $(16k - 2)\pi$ . The total curvature of  $\hat{h}(p)$  is therefore greater than  $(16k - 2)\pi$ . By Lemma 5.8, we may construct  $H$  as above with  $\text{tot}(H(s, p)) > (16k - 2)\pi > 4k\pi$  (for all  $s \in [0, 1]$  and all  $p \in \partial((\mathbb{D}^2)^{(k-1)})$ ) and therefore  $H$  is disjoint from  $\mathcal{M}_k$ , as claimed.

Let  $\mathbb{D}^{(2k-2)}$  be the closed disk of dimension  $2k - 2$  and radius 1. There exists a homeomorphism from  $\mathbb{D}^{(2k-2)}$  to  $(\{0\} \times (\mathbb{D}^2)^{(k-1)}) \cup ([0, 1] \times \partial((\mathbb{D}^2)^{(k-1)}))$ . Compose this homeomorphism with  $\hat{h}$  and  $H$  to define a map  $\tilde{h} : \mathbb{D}^{(2k-2)} \rightarrow \mathcal{L}_{(-1)^{k_z}}$  with  $\tilde{h}(0) \in \mathcal{M}_k$ ;  $\tilde{h}(p) \in \mathcal{M}_k$  implies  $p = 0$ ;  $\tilde{h}$  is topologically transversal to  $\mathcal{M}_k$ , with a single intersection at  $p = 0$ ;  $\text{tot}(\tilde{h}(p)) > 16k\pi$  for all  $p \in \mathbb{S}^{(2k-3)}$ . We furthermore have  $\tilde{h}(p) = \gamma^{[t_k \# 2]}$  for all  $p \in \mathbb{S}^{(2k-3)}$  (for some  $\gamma$ ). More precisely, let  $Q_k = \mathfrak{F}_{\gamma_k}(t_k)$ ; after a reparametrization we may assume that, for all  $p \in \mathbb{S}^{(2k-3)}$ ,  $(\tilde{h}(p))(t_k + \epsilon_1\tau) = Q_k\nu_2(\tau + \frac{1}{2})$  for  $|\tau| \leq \frac{1}{2}$ , where  $\epsilon_1 \in (0, \frac{1}{4k})$  is a small positive constant.

The sphere  $\mathbb{S}^{(2k-2)}$  is homeomorphic to  $S = (\{0, 1\} \times \mathbb{D}^{(2k-2)}) \cup ([0, 1] \times \mathbb{S}^{(2k-3)})$ . Let  $\gamma_2 : [0, 1] \rightarrow \mathcal{L}_1$  with  $\gamma_2(0) = \nu_2$ ,  $\gamma_2(1) = \nu_4$ . Define  $\tilde{h}_{2k-2} : S \rightarrow \mathcal{L}_{(-1)^{k_z}}$  by

$$\tilde{h}_{2k-2}(s, p)(t) = \begin{cases} (\tilde{h}(p))(t), & t \notin (t_k - \frac{\epsilon_1}{2}, t_k + \frac{\epsilon_1}{2}), \\ (\tilde{h}(p))(t), & s = 0, \\ (\tilde{h}(p))^{[t_k \# 2]}(t), & s = 1, \\ Q_k \gamma_2(s)(\tau + \frac{1}{2}), & |p| = 1, t = t_k + \epsilon_1 \tau, \tau \in [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

We claim that  $\tilde{h}_{2k-2}$  is homotopic to a point in  $\mathcal{I}_{(-1)^{k_z}}$ , or, equivalently (Proposition 5.6), that  $\tilde{h}_{2k-2}^{[t_k \# 2]}$  can be extended to a map from  $[0, 1] \times \mathbb{D}^{(2k-2)}$  to  $\mathcal{L}_{(-1)^{k_z}}$ . Indeed, after a reparametrization,  $\tilde{h}^{[t_k \# 2]} : \mathbb{D}^{(2k-2)} \rightarrow \mathcal{L}_{(-1)^{k_z}}$  may be assumed to satisfy  $(\tilde{h}^{[t_k \# 2]}(p))(t_k + \epsilon_1 \tau) = \tilde{Q}(p)\nu_2(\tau + \frac{1}{2})$  for all  $p \in \mathbb{D}^{(2k-2)}$ , where  $\tau \in [-\frac{1}{2}, \frac{1}{2}]$  and  $\tilde{Q} : \mathbb{D}^{(2k-2)} \rightarrow SO_3$  satisfies  $\tilde{Q}(p) = Q_k$  if  $|p| = 1$ . Define  $\bar{h} : [0, 1] \times \mathbb{D}^{(2k-2)} \rightarrow \mathcal{L}_{(-1)^{k_z}}$  by

$$\bar{h}(s, p)(t) = \begin{cases} (\tilde{h}^{[t_k \# 2]}(p))(t), & t \notin (t_k - \frac{\epsilon_1}{2}, t_k + \frac{\epsilon_1}{2}), \\ \tilde{Q}(p)\gamma_2(s)(\tau + \frac{1}{2}), & t = t_k + \epsilon_1 \tau, \tau \in [-\frac{1}{2}, \frac{1}{2}]; \end{cases}$$

up to reparametrization,  $\bar{h}$  is the desired extension.

Identifying  $S$  with  $\mathbb{S}^{2k-2}$ , the function  $\tilde{h}_{2k-2} : \mathbb{S}^{2k-2} \rightarrow \mathcal{L}_{(-1)^{k_z}}$  thus satisfies item (d). By construction, the only multiconvex curves in its image are  $\nu_k = \tilde{h}_{2k-2}(0, 0)$  and  $\tilde{h}_{2k-2}(1, 0)$ , which is a reparametrization of  $\nu_{k+2}$ . Define  $h_{2k-2}$  by perturbing  $\tilde{h}_{2k-2}$  near  $(1, 0)$  so as to avoid  $\mathcal{M}_{k+2}$ ; by transversality, this can be done: the codimension of  $\mathcal{M}_{k+2}$  is larger than the dimension of  $\mathbb{S}^{2k-2}$ . Item (b) is therefore satisfied. Topological transversality in item (a) also follows by construction and by items (c) and (f) of Lemma 7.2. Finally,  $m_{2k-2}(h_{2k-2}) = \pm 1$  follows from items (a) and (b), proving item (c). ■

The following corollary sums up some of the topological differences between the spaces  $\mathcal{L}_z$  and  $\mathcal{I}_z$  which we have proved in this section.

**Corollary 7.4** *Consider  $z \in \mathbb{S}^3$  with  $-z$  convex. For  $k \geq 1$ , the elements  $m_{2k-2} \in H^{2k-2}(\mathcal{L}_{(-1)^{k_z}})$  do not belong to the image of  $i^* = H^*(i) : H^*(\mathcal{I}_{(-1)^{k_z}}) \rightarrow H^*(\mathcal{L}_{(-1)^{k_z}})$ . The maps  $h_{2k-2} : \mathbb{S}^{2k-2} \rightarrow \mathcal{L}_{(-1)^{k_z}}$  define non-zero elements in the kernel of  $i_* = \pi_{2k-2}(i) : \pi_{2k-2}(\mathcal{L}_{(-1)^{k_z}}) \rightarrow \pi_{2k-2}(\mathcal{I}_{(-1)^{k_z}})$ .*

The aim of the rest of the paper is to prove that these are, in a sense, the *only* differences. Our final lemma in this section is an easy consequence of the previous results and will be used later.

**Lemma 7.5** *Let  $B$  be a compact manifold of dimension  $n + 1$  with boundary  $\partial B = K$ . Let  $f_0 : B \rightarrow \mathcal{L}_z$  be a continuous map with  $f_0|_K$  disjoint from all  $\mathcal{M}_k$ . Then there exists a continuous map  $f_1 : B \rightarrow \mathcal{L}_z$  with  $f_0|_K = f_1|_K$  and  $f_1$  disjoint from all  $\mathcal{M}_k$ .*

**Proof:** For each  $k$ , let  $V_k \subset \mathcal{L}_z$  be a closed tubular neighborhood of  $\mathcal{M}_k$ . We may assume the sets  $V_k$  to be disjoint and  $f_0|_K$  to be disjoint from the sets  $V_k$ . We may furthermore assume the map  $h_{2k-2} : \mathbb{S}^{2k-2} \rightarrow \mathcal{L}_z$  to be topologically transversal to  $V_k$  so that  $D_k = h_{2k-2}^{-1}(V_k) \subset \mathbb{S}^{2k-2}$  is a disk around the south pole. Since  $\mathcal{M}_k$  is a contractible Hilbert manifold,  $\mathcal{M}_k$  is homeomorphic to the Hilbert space  $\mathcal{H}$ . Let  $\mathbb{D}^{2k-2} \subset \mathbb{R}^{2k-2}$  be the closed unit ball of radius 1. Let  $\psi_k = (\psi_{k,1}, \psi_{k,2}) : V_k \rightarrow \mathcal{H} \times \mathbb{D}^{2k-2}$  be a homeomorphism taking  $\mathcal{M}_k$  to  $\mathcal{H} \times \{0\}$ ; we may assume that  $\psi_{k,1} \circ h_{2k-2}$  is constant equal to 0 and that  $\phi_k = \psi_{k,2} \circ h_{2k-2}$  is a homeomorphism between  $D_k$  and  $\mathbb{D}^{2k-2}$ . For  $c \in \mathbb{R}$  and  $\gamma \in V_k$ , let  $\mu(c, \gamma) \in V_k$  be such that  $\psi_{k,1}(\mu(c, \gamma)) = c\psi_{k,1}(\gamma)$  and  $\psi_{k,2}(\mu(c, \gamma)) = \psi_{k,2}(\gamma)$ .

Let  $\tilde{D}_k \subset D_k$  be the inverse image under  $\phi_k$  of the closed disk of radius  $1/2$ . Let  $\rho : \tilde{D}_k \rightarrow \mathbb{S}^{2k-2}$  be a continuous map coinciding with the identity on  $\partial\tilde{D}_k$  and avoiding the south pole. Define

$$f_1(p) = \begin{cases} f_0(p), & f_0(p) \notin \bigcup_k V_k, \\ \mu(2|\psi_{k,2}(f_0(p))| - 1, f_0(p)), & f_0(p) \in V_k, |\psi_{k,2}(f_0(p))| \geq \frac{1}{2}, \\ h_{2k-2}(\rho(\phi_k^{-1}(\psi_{k,2}(f_0(p))))), & f_0(p) \in V_k, |\psi_{k,2}(f_0(p))| \leq \frac{1}{2}. \end{cases}$$

The map  $f_1$  satisfies the required conditions. ■

## 8 Grafting

Subintervals  $[t_0, t_1] \subset (0, 1)$  will be called *arcs*: an arc  $[t_0, t_1]$  is *graftable* for a locally convex curve  $\gamma$  if there exists a projective transformation taking  $\gamma$  to  $\gamma_1$  and real numbers  $\theta_0, \theta_1, \phi_0, \phi_1$  with  $-\pi/2 < \phi_0 < 0 < \phi_1 < \pi/2$  and

$$\tilde{\mathfrak{F}}_{\gamma_1}(t_0) = e^{\theta_0 \mathbf{k}/2} e^{\phi_0 \mathbf{j}/2}, \quad \tilde{\mathfrak{F}}_{\gamma_1}(t_1) = e^{\theta_1 \mathbf{k}/2} e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}. \quad (2)$$

Recall that  $\Pi(e^{\theta \mathbf{k}/2})$  is a rotation by  $\theta$  around the  $z$  axis:

$$\Pi(e^{\theta \mathbf{k}/2}) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

similarly,  $\Pi(e^{\phi \mathbf{j}/2})$  is a rotation by  $\phi$  around the  $y$  axis.

Translating into a more geometric language, equation 2 says that  $\gamma_1$  is tangent at  $t_i$  to the circle  $z = -\sin(\phi_i)$ ; in both cases the orientation of  $\gamma_1$  is consistent with a locally convex orientation of the circles.

Notice that the existence of the desired projective transformation depends only on the value of  $\tilde{\mathfrak{F}}_{\gamma}(t_0; t_1)$ . A matrix  $Q \in SO_3$  is *graftable* if  $\tilde{\mathfrak{F}}_{\gamma}(t_0; t_1) = Q$  implies that the arc  $[t_0, t_1]$  is graftable for  $\gamma$ .

**Lemma 8.1** *Let  $Q \in SO_3$ :  $Q$  is graftable if and only if  $Q$  belongs to one of the Bruhat cells below:*

$$\text{Bru}_{(13);1}, \text{Bru}_{(13);4}, \text{Bru}_{(13);7}, \text{Bru}_{(123);3}, \text{Bru}_{(123);5}, \text{Bru}_{(132);5}, \text{Bru}_{(132);6}, \text{Bru}_{e;5}.$$

**Proof:** The definition is clearly invariant under projective transformation and therefore if  $Q_1$  and  $Q_2$  belong to the same Bruhat cell then  $Q_1$  is graftable if and only if  $Q_2$  is.

We need therefore to check which Bruhat cells are touched by

$$\mathfrak{F}_{\gamma_1}(t_0; t_1) = e^{-\phi_0 \mathbf{j}/2} e^{(\theta_1 - \theta_0) \mathbf{k}/2} e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}.$$

A straightforward computation (or a few figures) show that if  $\theta_0 = \theta_1$  then  $\mathfrak{F}_{\gamma_1}(t_0; t_1) = e^{(\phi_1 - \phi_0) \mathbf{j}/2} e^{\pi \mathbf{i}/2} \in \text{Bru}_{(13);7}$ . We may keep  $\theta_0$  fixed and change  $\theta_1$  and the Bruhat cell will cycle. If  $\phi_0 + \phi_1 < 0$  the cell will go through  $P_{(13);7}$ ,  $P_{(123);5}$ ,  $P_{(13);1}$  and  $P_{(123);3}$  (and back); notice that the cells  $\text{Bru}_{(13);7}$  and  $\text{Bru}_{(13);1}$  are open and correspond to intervals while the cells  $\text{Bru}_{(123);5}$  and  $\text{Bru}_{(123);3}$  have dimension 2 and correspond to transition points. If  $\phi_0 + \phi_1 > 0$  the cell will go through  $P_{(13);7}$ ,  $P_{(132);6}$ ,  $P_{(13);4}$  and  $P_{(132);5}$ . In the special case  $\phi_0 + \phi_1 = 0$  the only transition point is  $P_{e;5}$ . ■

We give a few examples which will be used again later:

$$\begin{aligned} Q_{0,1} &= \Pi(e^{\pi \mathbf{j}/6}), & Q_{1,1} &= \Pi(e^{\pi \mathbf{k}/2} e^{-\pi \mathbf{j}/12} e^{\pi \mathbf{i}/2}), \\ Q_{0,4} &= \Pi(e^{\pi \mathbf{j}/12}), & Q_{1,4} &= \Pi(e^{\pi \mathbf{k}/2} e^{-\pi \mathbf{j}/6} e^{\pi \mathbf{i}/2}), \\ Q_{0,7} &= \Pi(e^{\pi \mathbf{j}/8}), & Q_{1,7} &= \Pi(e^{-\pi \mathbf{j}/8} e^{\pi \mathbf{i}/2}). \end{aligned}$$

We clearly have that  $Q_{0,\ell}^{-1} Q_{1,\ell} \in \text{Bru}_{(13);\ell}$ ; also, if  $\mathfrak{F}_{\gamma_1}(t_i) = Q_{i,\ell}$  then  $\gamma_1$  satisfies the conditions in equation 2.

A minor difficulty is that the choices of  $\theta_0, \theta_1, \phi_0, \phi_1$  and of the projective transformation should be uniform. The following lemma addresses this issue.

**Lemma 8.2** *Let  $\ell$  be equal to 1, 4 or 7. Let  $Q_0, Q_1 \in SO_3$  be such that*

$$Q_0^{-1} Q_1 \in \text{Bru}_{(13);\ell}.$$

*Then there exists a unique matrix  $U \in Up_3^1$  such that  $\Pi(Q_{0,\ell} U Q_0^{-1})(Q_i) = Q_{i,\ell}$  for  $i = 0, 1$ .*

**Proof:** We have  $\Pi(Q_0^{-1})(Q_0) = I$  and  $\Pi(Q_0^{-1})(Q_1) = Q_0^{-1} Q_1 \in \text{Bru}_{(13);\ell}$ . There exists a unique  $U \in Up_3^1$  with  $\Pi(U)(Q_0^{-1} Q_1) = Q_{0,\ell}^{-1} Q_{1,\ell}$  and the result follows. ■

Given a curve  $\gamma \in \mathcal{L}$  and a graftable arc  $[t_0, t_1]$  such that  $\mathfrak{F}_\gamma(t_0; t_1) \in \text{Bru}_{(13);\ell}$  (where  $\ell$  equals 1, 4 or 7) we shall assume that  $\gamma_1 = \pi(Q_{0,\ell} U Q_0^{-1}) \circ \gamma$  where  $U \in Up_3^1$  is as in Lemma 8.2.

Consider a curve  $\gamma_1 \in \mathcal{L}$  and a graftable arc  $[t_0, t_1]$  such that equation 2 is satisfied. Given a real number  $s \geq 0$  we show how to perform a *graft* on the curve  $\gamma_1$  around the arc  $[t_0, t_1]$  to obtain a curve  $\gamma_1^{[(t_0, t_1) \# s]}$ . As usual, write  $\tilde{\Gamma}_1 = \tilde{\mathfrak{F}}_{\gamma_1}$ . Let  $\epsilon > 0$  be a small number. Define  $\tilde{\Gamma}_1^{[(t_0, t_1) \# s]}$  by  $\tilde{\Gamma}_1^{[(t_0, t_1) \# s]}(t) = \tilde{\Gamma}_1(t)$  for  $t \leq t_0$  and for  $t \geq t_1$ ; otherwise

$$\tilde{\Gamma}_1^{[(t_0, t_1) \# s]}(t) = \begin{cases} \exp\left(\left(\frac{\theta_0}{2} + \frac{\pi(t-t_0)}{\epsilon}\right) \mathbf{k}\right) e^{\phi_0 \mathbf{j}/2}, & t_0 \leq t \leq t_0 + s\epsilon, \\ e^{\pi s \mathbf{k}} \tilde{\Gamma}_1(2t - t_0 - 2s\epsilon), & t_0 + s\epsilon \leq t \leq t_0 + 2s\epsilon, \\ e^{\pi s \mathbf{k}} \tilde{\Gamma}_1(t), & t_0 + 2s\epsilon \leq t \leq t_1 - 2s\epsilon, \\ e^{\pi s \mathbf{k}} \tilde{\Gamma}_1(2t - t_1 + 2s\epsilon), & t_1 - 2s\epsilon \leq t \leq t_1 - s\epsilon, \\ \exp\left(\left(\frac{\theta_1}{2} + \frac{\pi(t_1-t)}{\epsilon}\right) \mathbf{k}\right) e^{\phi_1 \mathbf{j}/2} e^{\pi \mathbf{i}/2}, & t_1 - s\epsilon \leq t \leq t_1. \end{cases}$$

Equation 2 guarantees compatibility and continuity. Define (of course)

$$\gamma_1^{[(t_0, t_1) \# s]}(t) = \Pi(\tilde{\Gamma}_1^{[(t_0, t_1) \# s]}(t))e_1.$$

A geometric description of this construction is in order: the curve  $\gamma_1$  is cut at  $t_0$  and  $t_1$ , the arc from  $t_0$  to  $t_1$  is rotated by an angle of  $2\pi s$  around the  $z$  axis and finally arcs of circle (parallel to the plane  $z = 0$ ) are grafted into the resulting gap (see Figure 9). Notice that  $\tilde{\Gamma}_1^{[(t_0, t_1) \# s]}(t)$  is continuous as a function of  $s$  and  $t$ ; in other words, if  $\gamma_1 \in \mathcal{L}_Q$  then  $s \mapsto \gamma_1^{[(t_0, t_1) \# s]}$  is a continuous path from  $[0, s_{\max}]$  to  $\mathcal{L}_Q$ .

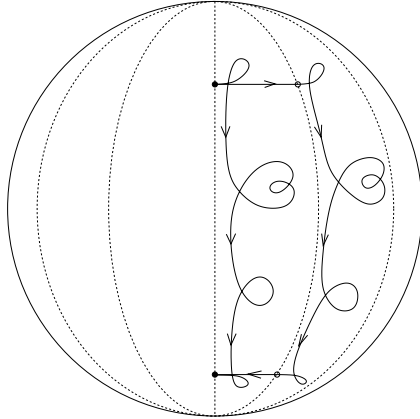


Figure 9: Grafting a curve

We now define grafting on  $\gamma$  provided  $\mathfrak{F}_\gamma(t_0; t_1) \in \text{Bru}_{(13);\ell}$  where  $\ell$  equals 1, 4 or 7. Let  $U$  be as in Lemma 8.2 and  $M = Q_{0,\ell} U Q_0^{-1} \in SL_3$  so that the projective

transformation  $\pi(M)$  takes  $\gamma$  to  $\gamma_1 = \pi(M) \circ \gamma$  satisfying  $\mathfrak{F}_{\gamma_1}(t_i) = Q_{i,\ell}$ . Next graft  $\gamma_1$  as above to obtain  $\gamma_1^{[(t_0, t_1) \# s]}$  and define  $\gamma^{[(t_0, t_1) \# s]} = \pi(M^{-1}) \circ \gamma_1^{[(t_0, t_1) \# s]}$ . Define a group homomorphism  $A : \mathbb{R} \rightarrow SL_3$  by  $A(s) = M^{-1} \Pi(e^{s\pi \mathbf{k}}) M = \exp(sa)$ ,  $a \in sl_3$ ; notice that  $A(1) = \exp(a) = I$ . We have

$$\gamma^{[(t_0, t_1) \# s]}(t) = \begin{cases} \gamma(t), & t \leq t_0, \\ \pi(\exp(sa))\gamma(t), & t_0 + 2s\epsilon \leq t \leq t_1 - 2s\epsilon, \\ \gamma(t), & t \geq t_1. \end{cases}$$

If  $s$  is a positive integer, the curves  $\gamma^{[(t_0, t_1) \# s]}$  (grafting) and  $\gamma^{[t_0 \# s; t_1 \# s]}$  (adding loops, as in Section 5, just before Lemma 5.1) are very similar. The only significant difference is that the “loops” in the first curve are closed convex curves which are not quite circles. Since the space of closed convex curves with a given base point is contractible these loops can easily be made round.

We write  $A \Subset B$  if the closure of  $A$  is contained in the interior of  $B$ . Notice that if  $A$  is open and closed then  $A \Subset A$ .

**Lemma 8.3** *Let  $\ell$  equal 1, 4 or 7. Let  $K$  be a compact manifold and  $f : K \rightarrow \mathcal{L}$  be a continuous map. Let  $K_0 \subset K$  be a compact subset. Assume that there exist continuous functions  $t_0 < t_1 : K_0 \rightarrow (0, 1)$  such that, for all  $p \in K_0$ ,  $\mathfrak{F}_{f(p)}(t_0; t_1) \in \text{Bru}_{(13);\ell}$ . Let  $W_0 \Subset K_0$  be an open set. Then, for sufficiently small  $\epsilon > 0$ , there exist a homotopy  $H : [0, 1] \times K \rightarrow \mathcal{L}$  and a function  $A : [0, 1] \times K \rightarrow SL_3$  with the following properties:*

- (a)  $t_0(p) + 8\epsilon < t_1(p) - 8\epsilon$  for all  $p \in K_0$ ;
- (b)  $A(p) = I$  for all  $p \in (K \setminus K_0) \cup W_0$ ;
- (c)  $H(0, p) = f(p)$  for all  $p \in K$ ;
- (d)  $H(s, p) = f(p)$  for all  $p \in K \setminus K_0$ ;
- (e)  $H(s, p)(t) = f(p)(t)$  for all  $p \in K_0$  and for all  $t \notin (t_0(p), t_1(p))$  and all  $s \in [0, 1]$ ;
- (f)  $H(s, p)(t) = \pi(A(s, p))(f(p)(t))$  for all  $p \in K_0$  and for all  $t \in (t_0(p) + 8\epsilon, t_1(p) - 8\epsilon)$  and all  $s \in [0, 1]$ ;
- (g)  $H(1, p) = f(p)^{[(t_0(p) + 4\epsilon) \# 2; (t_1(p) - 4\epsilon) \# 2]}$  for all  $p \in W_0$ .

**Proof:** Let  $W_1 \Subset K_0$  be an open set such that  $W_0 \Subset W_1$ . Let  $u : K \rightarrow [0, 1]$  be a smooth function with  $u(p) = 0$  for  $p \notin K_0$  and  $u(p) = 1$  for  $p \in W_1$ .

For  $s \in [0, 1/2]$ , define  $H(s, p)$  by grafting  $f(p)$ :

$$H(s, p) = f(p)^{[(t_0(p), t_1(p)) \# (4su(p))]}.$$

Notice that for  $p \in W_1$  we have  $H(1/2, p) = f(p)^{[(t_0(p), t_1(p))\#2]}$ . For  $s \in [1/2, 1]$  and  $p \notin W_1$  we define  $H(s, p) = H(1/2, p)$ . For  $p \in W_1$  we use the interval to round up the loops introduced by grafting. The margin  $W_1 \setminus W_0$  is needed for compatibility but for  $p \in W_0$  we may assume that  $H(1, p) = f(p)^{[(t_0(p)+4\epsilon)\#2; (t_1(p)-4\epsilon)\#2]}$ , completing our proof. ■

We are now ready to prove the easier cases of our main theorem; these are also proved in [20] using different ideas.

**Corollary 8.4** *Let  $z \in \mathbb{S}^3$  with  $\Pi(z) \in \text{Bru}_{(13);1} \cup \text{Bru}_{(13);4} \cup \text{Bru}_{(13);7}$ . Then the inclusion  $\mathcal{L}_z \subset \mathcal{I}_z$  is a weak homotopy equivalence.*

Since these spaces are Hilbert manifolds, they are actually diffeomorphic ([3]).

**Proof:** Let  $\ell$  be such that  $\Pi(z) \in \text{Bru}_{(13);\ell}$ . Let  $K$  be a compact manifold and  $f : K \rightarrow \mathcal{L}_z$  be a continuous map. For sufficiently small  $\epsilon > 0$  we have  $\mathfrak{F}_{f(p)}(\epsilon, 1 - \epsilon) \in \text{Bru}_{(13);\ell}$ . Apply Lemma 8.3 to  $f$  with  $t_0 = \epsilon$ ,  $t_1 = 1 - \epsilon$ ,  $K_0 = W_0 = K$  to deduce that  $f$  is homotopic to  $f^{[t_a\#2; t_b\#2]}$  and therefore loose. It now follows from Proposition 5.6 that  $f$  is homotopic to a constant in  $\mathcal{L}_z$  if and only if  $f$  is homotopic to a constant in  $\mathcal{I}_z$ . Together with Lemma 5.5, this completes the proof. ■

Part of Little's Theorem is that the set  $\mathcal{L}_{-1,c}$  of simple locally convex curves is a connected component of  $\mathcal{L}_{-1}$ : Fenchel proves that simple closed locally convex curves are convex ([8]; see also [15] and [24]); we often use this fact. The other part of Little's Theorem is that, once convex curves have been removed, the sets  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$  are path connected. This is again a corollary of Lemma 8.3.

**Corollary 8.5** *The sets  $\mathcal{L}_{+1}$  and  $\mathcal{L}_{-1,n}$  are path connected.*

**Proof:** Consider a map from  $\mathbb{S}^0$  (two points) to either of these spaces (i.e., two curves): the map is homotopic to a constant in  $\mathcal{I}_{\pm 1}$ . Each of the two curves  $\gamma_0$  and  $\gamma_1$  may be assumed generic and therefore to have a transversal self-intersection. As in figure 10, near the self-intersection there exist  $t_0$  and  $t_1$  such that  $\mathfrak{F}_\gamma(t_0; t_1) \in \text{Bru}_{(13);7}$  (there are other pairs for which this expression belongs to any of the other three open cells). ■

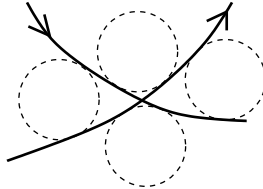


Figure 10: A transversal self-intersection



## 9 Good and bad steps

Given a locally convex curve  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$ , we now define the *next step* function  $\text{ns}_\gamma : [t_0, t_1^-] \rightarrow [t_0, t_1]$ , or, to make it shorter when  $\gamma$  is clear from the context,  $t^+ = \text{ns}_\gamma(t)$  (the number  $t_1^- \in [t_0, t_1]$  will also be defined). Given  $t \in [t_0, t_1]$  let  $t^+$  be the smallest  $\tilde{t} > t$  such that  $\Gamma(t; \tilde{t}) \notin \text{Bru}_{(13),2}$ ; if no such  $\tilde{t}$  exists then  $t^+$  is undefined. Since  $\text{Bru}_{(13),2}$  is an open set,  $\text{ns}_\gamma$  is a continuous function (where defined). Also, the function  $\text{ns}_\gamma$  is strictly increasing with continuous inverse, which will be denoted by  $t^- = \text{ns}_\gamma^{-1}(t)$ . Unless  $\gamma : [t_0, t_1] \rightarrow \mathbb{S}^2$  is convex,  $\text{ns}_\gamma$  is a strictly increasing continuous homeomorphism from  $[t_0, t_1^-]$  to  $[t_0^+, t_1]$ . On the other hand,  $\text{ns}_\gamma$  is usually not differentiable (even if  $\gamma$  is smooth).

Geometrically speaking,  $t_0^+$  is the point at which the arc  $\gamma|_{[t_0, t_0^+]}$  is still convex but about to somehow lose convexity. This can occur in five different ways corresponding to five Bruhat cells to which  $\Gamma(t_0; t_0^+)$  may belong. The two generic cases are when  $\gamma$  is about to leave the hemisphere defined by  $\Gamma(t_0)$  (but not at the point  $\gamma(t_0)$ ) or, conversely, when the geodesic defined by  $\Gamma(t)$  passes through  $\gamma(t_0)$  (but not aligned with  $\gamma'(t_0)$ ) so that  $\gamma$  is about to enter its own convex hull: these correspond to  $P_{(123);6}$  and  $P_{(132);0}$ , in this order, and to the first two diagrams in Figure 7: Notice that these matrices have two inversions and therefore their Bruhat cells have dimension 2. Two more exceptional cases correspond to the matrices  $P_{(23);2}$  and  $P_{(12);4}$  which, having one inversion, correspond to Bruhat cells of dimension 1. The two cases correspond to the third and fourth diagram in Figure 7: the curve may self-intersect by coming back to  $\gamma(t_0)$  (but with non-aligned tangent vectors) or it may tangentially touch the geodesic defined by  $\Gamma(t_0)$  (but not at  $\gamma(t_0)$ ). The fifth and most exceptional case corresponds to  $P_{e;0} = I$ , with Bruhat cell of dimension 0: the curve may come back to  $\gamma(t_0)$  with tangent vectors also aligned (as in the fifth diagram).

We are particularly interested in this fifth and most degenerate case. A *step* (for  $\gamma$ ) is an interval  $[t_0, t_1]$  with  $t_1 = t_0^+$ . A step is *bad* if  $\Gamma(t_1) = \Gamma(t_0)$  (this is the fifth case) and *good* otherwise. Notice that the set of good steps is open (in the space of steps). A curve  $\gamma$  is multiconvex if and only if, for  $t_0 = 0$ , the steps  $[t_0, t_1], [t_1, t_2], \dots$  are all bad (here  $t_j = \text{ns}_\gamma^j(t_0)$ ). Conversely, if a curve is complicated (i.e., not multiconvex) then, again with  $t_j = \text{ns}_\gamma^j(t_0)$ , there exists a good step  $[t_j, t_{j+1}] \subset [0, 1]$

A *good arc* for a locally convex curve  $\gamma$  is an interval  $[t_0, t_1] \subset (0, 1)$  such that:

- (a)  $t_0 < t_1^- < t_0^+ < t_1$ ;
- (b) if  $t \in [t_0, t_1^-]$  then  $[t, t^+]$  is a good step;
- (c) if  $t_0 \leq t_a < t_a^+ < t_b \leq t_1$  then  $\mathfrak{F}_\gamma(t_a; t_b)$  is in one of the following Bruhat cells:  $\text{Bru}_{(13);1}$ ,  $\text{Bru}_{(13);4}$ ,  $\text{Bru}_{(13);7}$ ,  $\text{Bru}_{(123);3}$ ,  $\text{Bru}_{(123);5}$ ,  $\text{Bru}_{(132);5}$  or  $\text{Bru}_{(132);6}$ .

The first three matrices in item (c) correspond of course to open cells as in Figure 6 above; the last four are shown in Figure 11.

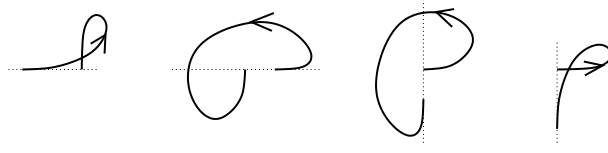


Figure 11: Good arcs

A *good pair of arcs* for a locally convex curve  $\gamma$  is a pair of good arcs  $[t_0, t_1] \subset [\tilde{t}_0, \tilde{t}_1] \subset (0, 1)$  such that  $\tilde{t}_0 < t_0 < t_1 < \tilde{t}_1$  and if  $\mathfrak{F}_\gamma(t_0; t_1) \notin \text{Bru}_{(13),1} \cup \text{Bru}_{(13),4}$  then  $\mathfrak{F}_\gamma(\tilde{t}_0; \tilde{t}_1) \in \text{Bru}_{(13),7}$ .

**Lemma 9.1** *Let  $K$  be a compact manifold; let  $f : K \rightarrow \mathcal{L}$  be a family of locally convex curves. Let  $t_0, t_1 : K \rightarrow (0, 1)$  be continuous functions with  $\text{ns}_{f(p)}(t_0(p)) = t_1(p)$  such that  $[t_0(p), t_1(p)]$  is always a good step.*

- (a) *There exists  $\epsilon > 0$  such that  $[t_0(p) - \epsilon, t_1(p) + \epsilon] \subset (0, 1)$  is always a good arc.*
- (b) *For any  $\epsilon > 0$  there exist continuous functions  $\tilde{t}_0 < \hat{t}_0 < \hat{t}_1 < \tilde{t}_1 : K \rightarrow (0, 1)$  with  $t_0 - \epsilon < \tilde{t}_0 < \hat{t}_0 < t_0$  and  $t_1 < \hat{t}_1 < \tilde{t}_1 < t_1 + \epsilon$  such that, for all  $p \in K$ ,  $[\hat{t}_0(p), \hat{t}_1(p)] \subset [\tilde{t}_0(p), \tilde{t}_1(p)]$  is a good pair of arcs for  $f(p)$ .*

**Proof:** Consider the union of all allowed cells in the definition of a good arc for  $\mathfrak{F}_\gamma(t_a; t_b)$  if  $t_a < t_a^+ < t_b$ :

$$A_1 = \text{Bru}_{(13),1} \cup \text{Bru}_{(13),4} \cup \text{Bru}_{(13),7} \cup \\ \cup \text{Bru}_{(123),3} \cup \text{Bru}_{(123),5} \cup \text{Bru}_{(132),5} \cup \text{Bru}_{(132),6} \subset SO_3.$$

Consider also the set of all allowed cells for  $\mathfrak{F}_\gamma(t_a; t_b)$  if  $t_a < t_b$ :

$$A_2 = A_1 \cup \text{Bru}_{(13),2} \cup \text{Bru}_{(123),6} \cup \text{Bru}_{(132),0} \cup \text{Bru}_{(23),2} \cup \text{Bru}_{(12),4} \subset SO_3.$$

We claim that the sets  $A_1$  and  $A_2$  are both open. For  $A_1$ , each of the four 2-cells included is sandwiched between two of the included 3-cells:  $\text{Bru}_{(123),3}$  between  $\text{Bru}_{(13),1}$  and  $\text{Bru}_{(13),7}$ ;  $\text{Bru}_{(123),5}$  between  $\text{Bru}_{(13),1}$  and  $\text{Bru}_{(13),7}$ ;  $\text{Bru}_{(132),5}$  between  $\text{Bru}_{(13),4}$  and  $\text{Bru}_{(13),7}$ ;  $\text{Bru}_{(132),6}$  between  $\text{Bru}_{(13),4}$  and  $\text{Bru}_{(13),7}$ . For  $A_2$ , since all four 3-dimensional cells are included we need only check that each of the two 1-dimensional cells included is surrounded by 2- and 3-dimensional cells which are also included:  $\text{Bru}_{(23),1}$  is surrounded by the four 3-dimensional cells plus  $\text{Bru}_{(132),0}$ ,  $\text{Bru}_{(123),6}$ ,  $\text{Bru}_{(123),3}$  and  $\text{Bru}_{(132),6}$ ;  $\text{Bru}_{(12),4}$  is surrounded by the four 3-dimensional cells plus  $\text{Bru}_{(123),6}$ ,  $\text{Bru}_{(132),0}$ ,  $\text{Bru}_{(132),5}$  and  $\text{Bru}_{(123),5}$ . This completes the proof of the claim

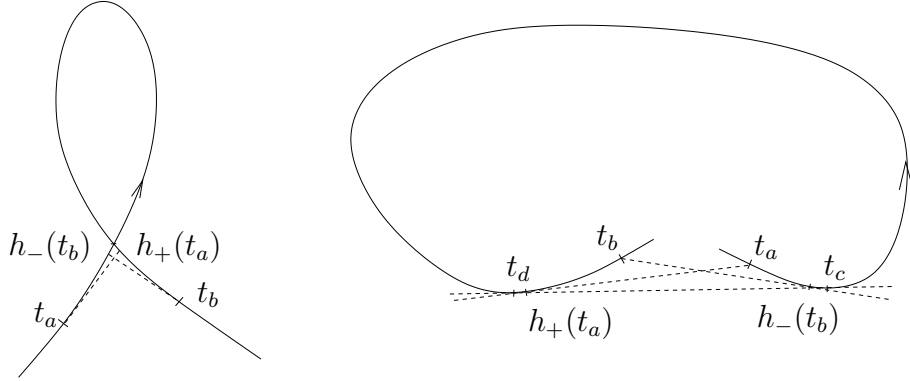
By hypothesis, if  $t_0(p) \leq t_a < t_b \leq t_1(p)$  then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_2$ . Thus, using the compactness of  $K$  and the fact that  $A_2$  is open, for sufficiently small  $\epsilon_2$ , if  $t_0(p) - \epsilon_2 < t_a < t_b < t_1(p) + \epsilon_2$  then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_2$ . In particular,  $[t_a, t_a^+]$  is a good step. For any good step  $[t_a, t_a^+]$  for a locally convex curve  $\gamma$  there exists  $\epsilon_a > 0$  such that if  $t_a^+ < t_b < t_a^+ + \epsilon_a$  then  $\mathfrak{F}_\gamma(t_a; t_b) \in A_1$ . Again by compactness, there exists therefore  $\epsilon \in (0, \epsilon_2)$  such that, for all  $p$ , if  $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$  then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in A_1$ , proving (a).

For item (b), assume arcs parametrized by a constant multiple (depending on  $p$  only) of arc length. We claim that for sufficiently small  $\epsilon_b > 0$ , we may take  $\tilde{t}_0(p) = t_0(p) - 2\epsilon_b$ ,  $\hat{t}_0(p) = t_0(p) - \epsilon_b$ ,  $\hat{t}_1(p) = t_1(p) + \epsilon_b$ ,  $\tilde{t}_1(p) = t_1(p) + 2\epsilon_b$ :  $[\hat{t}_0(p), \hat{t}_1(p)] \subset [\tilde{t}_0(p), \tilde{t}_1(p)]$  is a good pair of arcs for  $f(p)$ .

Let  $B = \text{Bru}_{(123);6} \cup \text{Bru}_{(132);0} \cup \text{Bru}_{(23);2} \cup \text{Bru}_{(12);4} \subset A_2 \subset SO_3$ ; the set  $B$  is a topological manifold of dimension 2 homeomorphic to  $\mathbb{S}^1 \times (0, 1)$ ; the subsets  $\text{Bru}_{(23);2}, \text{Bru}_{(12);4} \subset B$  are closed with disjoint neighborhoods  $B_2, B_4 \subset B$ , respectively. We may assume the closures of  $B_2$  and  $B_4$  in  $B$  to be disjoint. Let  $s : K \rightarrow B$  be defined by  $s(p) = \mathfrak{F}_{f(p)}(t_0(p); t_1(p))$ ; let  $U_2 = s^{-1}(B_2)$ ,  $U_4 = s^{-1}(B_4)$ . Define open sets  $U_6 \subseteq s^{-1}(\text{Bru}_{(123);6})$  and  $U_0 \subseteq s^{-1}(\text{Bru}_{(132);0})$ , so that the sets  $U_i$  form an open cover of  $K$ . For sufficiently small  $\epsilon > 0$ , if  $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$  and  $p \in U_6$  (resp.  $p \in U_0$ ) then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in \text{Bru}_{(13);4}$  (resp.  $\mathfrak{F}_{f(p)}(t_a; t_b) \in \text{Bru}_{(13);1}$ ). For  $p \in U_6 \cup U_0$ , therefore, pairs of arcs will be good. We must focus on  $p \in U_2$  and  $p \in U_4$ .

Assume  $p \in U_2$ . For small  $\epsilon > 0$ , we may assume that the arc  $[t_0(p) - \epsilon, t_1(p) + \epsilon]$  has at most one self intersection. Let  $V_2 \subset U_2$  be the open set of points  $p \in U_2 \subset K$  for which there exist  $t_c, t_d \in (t_0(p) - \epsilon, t_1(p) + \epsilon)$  with  $t_c < t_d$ ,  $f(p)(t_c) = f(p)(t_d)$  so that  $t_c^+ = t_d$  and  $\mathfrak{F}_{f(p)}(t_c; t_d) \in \text{Bru}_{(23);2}$ . By taking  $\epsilon > 0$  sufficiently small, we may assume that if  $p \in s^{-1}(\text{Bru}_{(123);6}) \setminus V_2$  and  $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$  then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in \text{Bru}_{(13);4}$ . Similarly, we may assume that if  $p \in s^{-1}(\text{Bru}_{(132);0}) \setminus V_2$  and  $t_0(p) - \epsilon < t_a < t_a^+ < t_b < t_1(p) + \epsilon$  then  $\mathfrak{F}_{f(p)}(t_a; t_b) \in \text{Bru}_{(13);1}$ .

If  $p \in V_2$  and  $t \geq t_c$  set  $h_+(t) = t^+$ ; if  $t < t_c$ , let  $\tilde{t} = h_+(t) \in (t_d, t_1 + \epsilon)$  be uniquely defined by  $\mathfrak{F}_{f(p)}(t; \tilde{t}) \in \text{Bru}_{(123);3}$ ; more geometrically, draw a tangent geodesic to  $\gamma$  at  $t$ : the curve  $\gamma$  intersects the geodesic at  $\tilde{t}$ . The function  $h_+ : [t_c - \delta, t_c + \delta] \rightarrow [t_d, t_d + \delta]$  is continuous, decreasing for  $t < t_c$  and increasing for  $t > t_c$  and satisfies  $h_+(t_c) = t_d$  and  $h'_+(t_c) = 0$ . Similarly, if  $t \leq t_d$  let  $h_-(t) = t^-$ ; otherwise, if  $t > t_d$ , let  $\tilde{t} = h_-(t) \in (t_0 - \epsilon, t_c)$  be defined by  $\mathfrak{F}_{f(p)}(\tilde{t}; t) \in \text{Bru}_{(132);6}$ ; again, the function  $h_- : [t_d - \delta, t_d + \delta] \rightarrow [t_c - \delta, t_c]$  is continuous, increasing for  $t < t_d$  and decreasing for  $t > t_d$  and satisfies  $h_-(t_d) = t_c$  and  $h'_-(t_d) = 0$  (see Figure 12). By taking  $\epsilon$  small we may assume that the functions  $h_+$  and  $h_-$  are always  $(1/2)$ -Lipschitz, i.e., that  $|h_+(t_a) - h_+(\tilde{t}_a)| \leq |t_a - \tilde{t}_a|/2$  and  $|h_-(t_b) - h_-(\tilde{t}_b)| \leq |t_b - \tilde{t}_b|/2$ .

Figure 12: The functions  $h_+$  and  $h_-$ 

If  $p \in V_2$ ,  $t_0 - \epsilon < t_a < t_a^+ < t_b < t_1 + \epsilon$  and  $t_a > h_-(t_b)$  then  $\mathfrak{F}_\gamma(t_a; t_b) \in \text{Bru}_{(13),4}$ . Similarly, if  $t_b < h_+(t_a)$  then  $\mathfrak{F}_\gamma(t_a; t_b) \in \text{Bru}_{(13),1}$ . Thus, if  $\mathfrak{F}_\gamma(t_a; t_b) \notin \text{Bru}_{(13),1} \cup \text{Bru}_{(13),4}$  then  $t_a \leq h_-(t_b) \leq t_c < t_d \leq h_+(t_a) \leq t_b$ . Conversely, if  $t_a < h_-(t_b) < t_c < t_d < h_+(t_a) < t_b$  then  $\mathfrak{F}_\gamma(t_a; t_b) \in \text{Bru}_{(13),7}$ . But the Lipschitz condition means that if  $t_0 - \epsilon < t_a - \epsilon_b < t_a < t_a^+ < t_b < t_b + \epsilon_b < t_1 + \epsilon$  and  $\mathfrak{F}_\gamma(t_a; t_b) \notin \text{Bru}_{(13),1} \cup \text{Bru}_{(13),4}$  then  $\mathfrak{F}_\gamma(t_a - \epsilon_b; t_b + \epsilon_b) \in \text{Bru}_{(13),7}$ .

A similar construction holds in  $U_4$ , only the Bruhat classes change and the geometric construction of  $h_\pm$  is different (see Figure 12): to define  $h_+(t_a)$  we construct a tangent line to  $\gamma$  which passes through  $t_a$ . Alternatively, we apply Arnold duality ([2], [20]) to go from the  $U_4$  to the  $U_2$  scenario. ■

## 10 Complicated curves

Recall that a curve  $\gamma \in \mathcal{L}_I$  is *complicated* if it belongs to

$$\mathcal{Y}_Q = \mathcal{L}_Q \setminus \bigcup_k \mathcal{M}_k.$$

**Lemma 10.1** *Let  $K$  be a compact manifold and let  $f : K \rightarrow \mathcal{L}_Q$  be a continuous map. If the image of  $f$  is contained in  $\mathcal{Y}_Q$  then  $f$  is loose.*

**Proof:** Let  $\tilde{U}_0 \subseteq K$  be the open set of elements  $p \in K$  for which  $[0, 0^+]$  is a good step. More generally, let  $\tilde{U}_j \subseteq K$  be the open set of elements  $p \in K$  for which  $\text{ns}_{f(p)}^{j+1}(0) < 1$  and  $[\text{ns}_{f(p)}^j(0), \text{ns}_{f(p)}^{j+1}(0)]$  is a good step. Since all curves  $f(p)$  are complicated, the sets  $\tilde{U}_j$  cover  $K$ . By compactness of  $K$ , there exists an integer  $J$  such that  $K = \bigcup_{j < J} \tilde{U}_j$ . Define compact sets  $K_j \subset \tilde{U}_j$  such that the interiors  $U_j$  of  $K_j$  already cover  $K$ .

Again by compactness, there exists  $\epsilon_{J-1} > 0$  such that if  $p \in K_{J-1}$  and  $|t_{J-1} - \text{ns}_{f(p)}^{J-1}(0)| < \epsilon_{J-1}$  then  $\text{ns}_{f(p)}(t_{J-1}) < 1$  and  $[t_{J-1}, \text{ns}_{f(p)}(t_{J-1})]$  is a good step (for  $f(p)$ ). Define the continuous function  $t_{J-1} : K_{J-1} \rightarrow [0, 1]$  by  $t_{J-1}(p) = \epsilon_{J-1}/2 + \text{ns}_{f(p)}^{J-1}(0)$ . Similarly, there exists  $\epsilon_{J-2} > 0$  such that if  $p \in K_{J-2}$  and  $|t_{J-2} - \text{ns}_{f(p)}^{J-2}(0)| < \epsilon_{J-2}$  then  $\text{ns}_{f(p)}(t_{J-2}) < 1$ ,  $\text{ns}_{f(p)}(t_{J-2}) < t_{J-1}(p)$  if  $p \in K_{J-1}$  and  $[t_{J-2}, \text{ns}_{f(p)}(t_{J-2})]$  is a good step. Proceed in this manner so that we have continuous functions  $t_j : K_j \rightarrow (0, 1)$ ,  $0 \leq j < J$ , such that if  $p \in K_j$  then  $[t_j(p), \text{ns}_{f(p)}(t_j(p))] \subset (0, 1)$  is a good step for  $f(p)$  and if  $p \in K_{j_1} \cap K_{j_2}$ ,  $j_1 < j_2$ , then  $\text{ns}_{f(p)}(t_{j_1}(p)) < t_{j_2}(p)$ . Again by compactness, there exists  $\epsilon > 0$  such that  $t_j(p) > \epsilon$ ,  $\text{ns}_{f(p)} < 1 - \epsilon$  and if  $p \in K_{j_1} \cap K_{j_2}$ ,  $j_1 < j_2$ , then  $\text{ns}_{f(p)}(t_{j_1}(p)) + \epsilon < t_{j_2}(p)$ .

Thus if  $p \in K_j$  then the interior of the interval  $\hat{I}_j = [t_j(p) - \epsilon/3, \text{ns}_{f(p)}(t_j(p)) + \epsilon/3]$  contains the good step  $[t_j(p), \text{ns}_{f(p)}(t_j(p))]$ . Notice furthermore that for given  $p$  the intervals  $\hat{I}_j$  are disjoint. From Lemma 9.1 we can now define functions  $a_j < b_j < c_j < d_j : K \rightarrow (0, 1)$  such that, for all  $p \in K$ ,  $a_j(p), b_j(p), c_j(p), d_j(p) \in \hat{I}_j(p)$  and, for  $p \in K_j$ ,  $[b_j(p), c_j(p)] \subset [a_j(p), d_j(p)]$  is a good pair of arcs. Define  $I_j(p) = [b_j(p), c_j(p)]$  and  $\tilde{I}_j(p) = [a_j(p), d_j(p)]$ .

The strategy now is to deform curves in each interval  $I_j$  independently. More precisely, define

$$\begin{aligned} U_{j,1} &= \{p \in K_j; \mathfrak{F}_{f(p)}(b_j(p); c_j(p)) \in \text{Bru}_{(13),1}\}, \\ U_{j,4} &= \{p \in K_j; \mathfrak{F}_{f(p)}(b_j(p); c_j(p)) \in \text{Bru}_{(13),4}\}, \\ U_{j,7} &= \{p \in K_j; \mathfrak{F}_{f(p)}(a_j(p); d_j(p)) \in \text{Bru}_{(13),7}\}. \end{aligned}$$

By definition of good pair of arcs,  $U_j = U_{j,1} \cup U_{j,4} \cup U_{j,7}$ . Consider open sets  $W_{j,\ell}$  and compact sets  $K_{j,\ell}$  such that  $W_{j,\ell} \Subset K_{j,\ell} \Subset U_{j,\ell}$  and such that the sets  $W_{j,\ell}$  cover  $K$ .

Let  $f_0 = f$ ; apply Lemma 8.3 (with  $\ell = 1$ ,  $K_0 = K_{0,1}$ ,  $W_0 = W_{0,1}$ ,  $t_0 = b_0$  and  $t_1 = c_0$ ) to define a homotopy from  $f_0$  to another function  $f_{0,1}$  with  $f_{0,1}(p) = f_0(p)^{[b_0\#2;c_0\#2]}$  for all  $p \in W_{0,1}$ . Apply the same lemma (now with  $\ell = 4$ ,  $K_0 = K_{0,4}$ ,  $W_0 = W_{0,4}$ ,  $t_0 = b_1$  and  $t_1 = c_1$ ) to define a homotopy from  $f_{0,1}$  to  $f_{0,4}$  with  $f_{0,4}(p) = f_{0,1}(p)^{[b_0\#2;c_0\#2]}$  for all  $p \in W_{0,4}$ ; notice that since  $K_{0,1}$  and  $K_{0,4}$  are disjoint the two constructions do not interfere with one another. Apply Lemma 8.3 yet another time (now with  $\ell = 7$ ,  $K_0 = K_{0,7}$ ,  $W_0 = W_{0,7}$ ,  $t_0 = a_1$  and  $t_1 = d_1$ ) to define a homotopy from  $f_{0,4}$  to  $f_{0,7}$  with  $f_{0,7}(p) = f_{0,4}(p)^{[a_0\#2;d_0\#2]}$  for all  $p \in W_{0,7}$ ; notice that  $\mathfrak{F}_{f_{0,4}(p)}(a_0(p)) = \mathfrak{F}_{f(p)}(a_0(p))$  and  $\mathfrak{F}_{f_{0,4}(p)}(d_0(p)) = \mathfrak{F}_{f(p)}(d_0(p))$ . Even though the interval  $(a_0, d_0)$  contains the points  $b_0$  and  $c_0$ , the loops created in the two first steps were not destroyed but merely pushed around by a projective transformation. We may therefore define a homotopy from  $f_{0,7}$  to  $f_1$  such that if  $p \in W_{0,1} \cup W_{0,4}$  then  $f_1(p) = \gamma^{[b_0\#2;c_0\#2]}$  for some  $\gamma \in \mathcal{L}_Q$ ; if  $p \in W_{0,7}$  then  $f_1(p) = \gamma^{[a_0\#2;d_0\#2]}$  for some  $\gamma \in \mathcal{L}_Q$ .

Repeat the process to define a homotopy from  $f_1$  to  $f_2$  and so on until  $f_J$  such that, finally, for  $f_J$  we have:

- if  $p \in W_{j,1} \cup W_{j,4}$  then  $f_J(p) = \gamma^{[b_j \# 2; c_j \# 2]}$  for some  $\gamma \in \mathcal{L}_Q$ ;
- if  $p \in W_{j,7}$  then  $f_J(p) = \gamma^{[a_j \# 2; d_j \# 2]}$  for some  $\gamma \in \mathcal{L}_Q$ .

Since the sets  $W_{j,\ell}$  cover  $K$  we have from Lemma 5.8 that  $f_J$  is loose.  $\blacksquare$

We now prove that the inclusions  $\mathcal{Y}_z \subset \mathcal{I}_z$  are weak homotopy equivalences.

**Proof of Proposition 1.2:** Let  $K$  be a compact manifold of dimension  $n$  and let  $f : K \rightarrow \mathcal{I}_z$  a continuous map. From Lemma 5.5, for sufficiently large  $m$ ,  $f$  is homotopic to  $f_1 = F_{2m} \circ f$  and the image of  $f_1$  is contained in  $\mathcal{L}_z$ . Furthermore, the total curvature of  $f_1(p) = (F_{2m} \circ f)(p)$  tends to infinity when  $m$  tends to infinity. We may therefore choose  $m$  large enough so that  $\text{tot}(f_1(p)) > 8(n+1)\pi$  for all  $p \in K$ . In particular, the image of  $f_1$  is disjoint from  $\mathcal{M}_k$  for all  $k \leq n+1$ . For  $k > n+1$ , the codimension of  $\mathcal{M}_k$  equals  $2k-2 > 2n \geq n$ : by transversality, we may perturb  $f_1$  to define a homotopic map  $f_2 : K \rightarrow \mathcal{L}_z$  whose image is disjoint from all submanifolds  $\mathcal{M}_k$ , so that the image of  $f_2$  is contained in  $\mathcal{Y}_z$ . In particular the maps  $i_* = \pi_n(i) : \pi_n(\mathcal{Y}_z) \rightarrow \pi_n(\mathcal{I}_z)$  are surjective.

Conversely, let  $B$  be a compact manifold of dimension  $n+1$  with boundary  $K = \partial B$ . Let  $g : B \rightarrow \mathcal{I}_z$  be a continuous map with the image of  $f = g|_K$  contained in  $\mathcal{Y}_z$ . We prove that there exists a map  $\tilde{g} : B \rightarrow \mathcal{Y}_z$  with  $\tilde{g}|_K = f$ . Indeed, let  $g_1 = F_{2m} \circ g : B \rightarrow \mathcal{I}_z$ . Again from Lemma 5.5, for sufficiently large  $m$  we have that the image of  $g_1$  is contained in  $\mathcal{L}_z$ . From Lemma 10.1,  $f : K \rightarrow \mathcal{L}_z$  is homotopic (in  $\mathcal{L}_z$ ) to  $f_1 = g_1|_K$ . We therefore obtain a map  $g_2 : B \rightarrow \mathcal{L}_z$  with  $g_2|_K = f$ . By Lemma 7.5, there exists  $g_3 : B \rightarrow \mathcal{Y}_z \subset \mathcal{L}_z$  with  $g_3|_K = f$ . In particular the maps  $i_* = \pi_n(i) : \pi_n(\mathcal{Y}_z) \rightarrow \pi_n(\mathcal{I}_z)$  are injective.  $\blacksquare$

## 11 Proof of Theorems 1 and 2 and final remarks

We now have all the tools required to complete the proof of Theorem 2; Theorem 1 is then a special case.

**Proof of Theorem 2:** Recall that  $\mathcal{M}_k$  is either empty or a contractible submanifold of codimension  $2k-2$ . As in the proof of Lemma 7.5, for each  $k$ , if  $\mathcal{M}_k \neq \emptyset$  let  $V_k \subset \mathcal{L}_z$  be a closed tubular neighborhood of  $\mathcal{M}_k$  with interior  $U_k \subset V_k$ . If  $\mathcal{M}_k = \emptyset$ , let  $U_k = V_k = \emptyset$ ; also, for  $k=1$  if  $\mathcal{M}_k \neq \emptyset$  then  $\mathcal{M}_k$  is a contractible connected component and  $U_k = V_k = \mathcal{M}_k$ . As before, assume the sets  $V_k$  to be disjoint. If  $\mathcal{M}_k \neq \emptyset$ , let  $\psi_k = (\psi_{k,1}, \psi_{k,2}) : V_k \rightarrow \mathcal{H} \times \mathbb{D}^{2k-2}$  be a homeomorphism ( $\mathcal{H}$  is the Hilbert space). Again, assume that  $\mathcal{M}_k = \psi_{k,2}^{-1}(\{0\})$ . Assume furthermore that  $D_k = h_{2k-2}^{-1}(V_k) \subset \mathbb{S}^{2k-2}$  is a closed disk with smooth boundary. Assume also that  $\psi_{k,1} \circ h_{2k-2}$  is constant equal to 0 in  $D_k$  and that

$h_{2k-2}$  is a homeomorphism from  $D_k$  to  $\mathcal{D}_k = h_{2k-2}(D_k) = \psi_{k,1}^{-1}(\{0\}) \subset V_k$ . Let  $\tilde{\mathcal{Y}}_z \subset \mathcal{Y}_z$  and  $\tilde{\mathcal{L}}_z \subset \mathcal{L}_z$  be defined by

$$\tilde{\mathcal{Y}}_z = \mathcal{L}_z \setminus \bigcup_{\mathcal{M}_k \neq \emptyset} U_k, \quad \tilde{\mathcal{L}}_z = \tilde{\mathcal{Y}}_z \cup \bigcup_{\mathcal{M}_k \neq \emptyset} \mathcal{D}_k.$$

Since  $\mathcal{H} \times \mathbb{S}^{2k-3} \subset \mathcal{H} \times (\mathbb{D}^{2k-2} \setminus \{0\})$  is a deformation retract it follows that so is  $\tilde{\mathcal{Y}}_z \subset \mathcal{Y}_z$ . It thus follows from Proposition 1.2 that  $\tilde{\mathcal{Y}}_z \approx \mathcal{I}_z \approx \Omega\mathbb{S}^3$ . Similarly, since  $(\mathcal{H} \times \mathbb{S}^{2k-3}) \cup (\{0\} \times \mathbb{D}^{2k-2}) \subset \mathcal{H} \times \mathbb{D}^{2k-2}$  is a deformation retract, so is  $\tilde{\mathcal{L}}_z \subset \mathcal{L}_z$ .

If neither  $z$  nor  $-z$  is convex,  $\mathcal{M}_k = \emptyset$  for all  $k$  and  $\tilde{\mathcal{Y}}_z = \tilde{\mathcal{L}}_z$  and we are done. If  $z$  (resp.  $-z$ ) is convex then  $\mathcal{M}_k \neq \emptyset$  for  $k$  odd (resp. even). Thus, for  $z$  convex,  $\tilde{\mathcal{L}}_z$  is obtained from  $\tilde{\mathcal{Y}}_z \approx \Omega\mathbb{S}^3$  by gluing disks  $\mathbb{D}^0$  (i.e., adding a contractible connected component),  $\mathbb{D}^4$ ,  $\mathbb{D}^8$ ,  $\dots$ . Similarly, for  $-z$  convex,  $\tilde{\mathcal{L}}_z$  is obtained from  $\tilde{\mathcal{Y}}_z \approx \Omega\mathbb{S}^3$  by gluing disks  $\mathbb{D}^2$ ,  $\mathbb{D}^6$ ,  $\mathbb{D}^{10}$ ,  $\dots$ . The maps  $h_{2k-2}$  guarantee that the spheres along which these disks are being glued are nullhomotopic in  $\tilde{\mathcal{Y}}_z$  and therefore gluing a disk is (homotopically) equivalent to gluing a sphere: the theorem follows.  $\blacksquare$

The question of how the spaces  $\mathcal{L}_z$  fit together still requires some clarification. It should be noted that the map from  $\mathcal{L}$  to  $\mathbb{S}^3$  taking  $\gamma$  to  $\tilde{\mathfrak{F}}_\gamma(1)$  does not satisfy the homotopy lifting property (see also [18]). In particular, we would like to gain a better understanding of periodic solutions of a linear ODE of order 3.

Finally, similar questions can be asked about curves in  $\mathbb{S}^n$ ,  $n > 2$  ( $\gamma$  is locally convex if  $\det(\gamma(t), \dots, \gamma^{(n)}(t)) > 0$ ); in [20] we show a few results about these spaces. It would also be interesting to investigate the homotopy type of spaces of curves with bounded geodesic curvature.

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Nicolau C. Saldanha, PUC-Rio

saldanha@puc-rio.br; <http://www.mat.puc-rio.br/~nicolau/>

Departamento de Matemática, PUC-Rio

R. Marquês de S. Vicente 225, Rio de Janeiro, RJ 22453-900, Brazil